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Bankruptcy in Long-term Investments

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Long-term portfolio management is an important issue in modern finance and practice. We have analyzed various known continuous-time strategies in portfolio management, with a focus on bankruptcy probabilities under these strategies. We show that for each strategy, there is a threshold in the target return rate. When the target return rate is set above this threshold, the application of the strategy for a long investment horizon leads to sure bankruptcy. For a target return rate lower than this threshold, the bankruptcy never occurs. Bankruptcy probabilities under a finite investment horizon are also studied. An empirical study based on the Dow Jones Industrial Average Index confirms these results. By comparing behavior of these strategies in various parameter regions, we reveal connections among these seemingly different strategies.

Key words: Bankruptcy probability; Long-term investment; Threshold in expected return; Optimal portfolios
History:

1. Introduction

In recent years there have been several continuous-time optimal dynamic strategies designed for attaining various goals over an investment horizon. The comparison study among these continuous-time strategies and the issue of bankruptcy probability in the application of these strategies over a very long investment horizon have not yet received much attention. In this paper we will consider several known strategies in portfolio management and study these issues.

Karatzas (1997) shows that the portfolio selection criterions can be classified into two classes, depending on whether there is a lower bound in wealth. The processes with lower bound in wealth belong to the class of tamed portfolio processes, whereas the ones without lower bound in wealth belong to the class of untamed portfolio processes. Dybvig and Huang (1988) mentioned that a strategy which guarantees the portfolio value above a lower bound at all times can preclude arbitrage opportunities in the market, while, some strategies with no lower bound restriction may admit arbitrage opportunities, and hence become ill-defined in the continuous-time limit. Harrison and Kreps (1979) showed that without any constraints the untamed strategies may lead to arbitrage profits which can be obtained by following a so-called doubling strategy. However, imposing certain regularity constraints, such as square integrability of the trading strategies can eliminate such arbitrage opportunities.

Among untamed strategies, the most famous one is the Mean-Variance portfolio selection, which was first proposed in the single-period setting by Markowitz (1952, 1959) in his pioneering work. This was then generalized to multi-period setting by Hakansson (1971), Grauer and Hakansson (1993), and to continuous-time setting by Zhou and Li (2000). Li and Zhou (2007) have shown that the strategy of the mean-variance efficient portfolio can have a probability higher than 80% to achieve the target before the terminal date. This is a very interesting result. However, they also stated that there could be an equally high chance that one gets bankruptcy before reaching the target. Some other strategies in this class are Mean Quadratic Variation strategy studied by Brugiere (1996), Final Wealth Targeting strategy by Duffie and Richardson (1991), and Wealth Tracking strategy by Huang and Zhang (2006). Unfortunately, we find that long-term investments following these untamed strategies will surely lead to bankruptcy no matter how small the target return rate set. The reason is that strategy in the untamed class does not need to maintain any minimum value of the portfolio. When the value of a portfolio is small, in order to reach the target return rate, one has to borrow more money from the money market account to invest in the risk asset. This certainly increases the risk of the portfolio and, in consequence, dramatically increases the probability of bankruptcy. This is clearly not a practical approach. Therefore, we will not discuss these untamed strategies in this paper.

The main focus of this paper is on the three tamed strategies. The first is the well-known Power Utility Maximization (PUM) approach, introduced by Merton (1971) in his famous work. The second strategy was studied by Bielecki et al. (2005), which imposes a nonnegative-wealth restriction to the mean-variance approach. We call this the Modified Mean-Variance (MMV) strategy. The third strategy was studied by Browne (1999, 2000), which aims to minimize the probability of the portfolio value falling below a specified wealth target at a given investment horizon. We call this the Shortfall Probability Minimization (SPM) strategy.

All of these three strategies have the properties that the portfolio value will always remain nonnegative. One may wonder how bankruptcy can occur when the portfolio value is explicitly bounded below by zero. The answer is that the wealth can exactly be zero or arbitrarily close to zero, when one applies these three tamed strategies. Our main result shows that for long-term investments, each strategy has a unique threshold in the target return rate. In the infinite investment horizon limit, the bankruptcy probability is either zero or one, depending on whether the target return rate is set below or above this threshold.

We also study the bankruptcy probability over the finite time horizon. We show that for the PUM strategy, the bankruptcy probability is an increasing function of the investment horizon. This property is also true for the MMV and the SPM strategies when the target return rate is set above the threshold. However, when the target return rate is lower than the threshold, the latter two strategies (MMV and SPM) exhibit a maximum bankruptcy probability at a particular investment horizon. Furthermore, for all three tamed strategies, the maximum bankruptcy probability is a monotone increasing function of the target return rate.

We study these three tamed strategies from various points of view, including utility maximization perspective with certain utility functions. Our analysis reveals the close relationship among these three seemingly different strategies.

The model for a financial market in this paper is the same as that in Merton (1971), which consists of n log-normal distributed risky assets governed by

$$dS_i(t) = S_i(t)\left[\mu_i\, dt + \sum_{j=1}^{n} \sigma_{ij}\, dW_t^{(j)}\right], \quad i = 1, ..., n \tag{1}$$

and a money market with a constant risk-free interest rate $\tilde{r}$. Here μ_i and σ_{ij} are growth rate and volatility, respectively, and $\boldsymbol{W_t} \triangleq (W_t^{(1)}, ..., W_t^{(n)})'$ is a standard n-dimensional Brownian motion. We will show later in the paper that for all aforementioned strategies, the n-risky-assets problem is equivalent to a one-risky-asset problem, which is essentially the well-known two-fund separation theorem studied by Khanna and Kulldorff (1999). Therefore, for the sake of simplicity, most of the presentations below will be based on the case of a single risky asset, which follows the process

$$dS(\tilde{t}) = S(\tilde{t})(\tilde{\mu} d\tilde{t} + \tilde{\sigma} dW(\tilde{t})), \tag{2}$$

where $\tilde{\mu}$ and $\tilde{\sigma}$ are constants, $\tilde{\mu} > \tilde{r}$, $\tilde{\sigma} > 0$, and $W(\tilde{t})$ is a standard Brownian motion.

Let $X(\tilde{t})$ denote the present value of investor's wealth at time $\tilde{t}$ which is discounted by the factor $\exp(-\tilde{r}\tilde{t})$, and $\tilde{\pi}(\tilde{t})$ be the total wealth invested in the risky asset at time $\tilde{t}$. Then the discounted wealth process $X(\cdot)$ obeys:

$$\begin{aligned} dX(\tilde{t}) &= \tilde{\pi}(\tilde{t})(\tilde{\mu} - \tilde{r})d\tilde{t} + \tilde{\pi}(\tilde{t})\tilde{\sigma} dW(\tilde{t}), \\ X(0) &= x. \end{aligned} \tag{3}$$

We use the symbol "˜" to indicate that these quantities are dimensional, i.e., the ones with units; conversions to dimensionless quantities will be discussed shortly. In terms of these dimensional quantities, optimal portfolio selection problems for the three tamed strategies mentioned earlier can be expressed as follows:

1. Modified Mean-Variance (MMV) Problem:

$$\begin{aligned} \text{Goal: } \min_{\tilde{\pi}} \; & Var(X(\tilde{T})) \\ s.t. \;\; & E(X(\tilde{T})) = xe^{(\tilde{R}-\tilde{r})\tilde{T}}, \\ & X(\tilde{t}) \geqslant 0 \;\; a.s., \;\; \forall \tilde{t} \in [0, \tilde{T}]. \end{aligned}$$

2. Shortfall Probability Minimization (SPM) Problem:

$$\text{Goal: } \min_{\tilde{\pi}} \; P\left[X(\tilde{T}) < xe^{(\tilde{R}-\tilde{r})\tilde{T}}\right].$$

3. Power Utility Maximization (PUM) Problem:

$$\text{Goal: } \max_{\tilde{\pi}} \; E\left[\frac{1}{\gamma}(X(\tilde{T}))^{\gamma}\right].$$

where the risk aversion parameter γ is related to the target return rate $\tilde{R}$ (see Eq. (32)).

The quantity $\tilde{R}$ in the aforementioned formulations is defined to be the undiscounted dimensional target return rate for the investment horizon $\tilde{T}$, which is a free-varying parameter for each strategy. Different values of $\tilde{R}$ lead to a different wealth allocation to the risky asset for each given strategy. Therefore each tamed strategy (MMV, SPM and PUM) is actually a one-parameter family of portfolio choices.

We introduce the following dimensionless quantities:

$$R \triangleq \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^{-2} (\tilde{R}-\tilde{r}), \tag{4}$$

$$t \triangleq \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^{2} \tilde{t}, \tag{5}$$

and a scaled quantity:

$$\pi(\cdot) \triangleq \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}^2}\right)^{-1} \tilde{\pi}(\cdot). \tag{6}$$

Unless otherwise specified, dimensionless and scaled quantities will be used in the rest of this paper. The corresponding unscaled dimensional results can easily be recovered from Eqs. (4), (5) and (6). More specifically, for given dimensional parameters $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{r}$, the undiscounted dimensional $\tilde{R}$, the dimensional $\tilde{t}$ and $\tilde{\pi}(\tilde{t},\tilde{R})$ can be obtained from the dimensionless R, t and $\pi(t,R)$ through the relations:

$$\tilde{R} = \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^2 R + \tilde{r}, \tag{7}$$

$$\tilde{t} = \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^{-2} t, \tag{8}$$

$$\tilde{\pi}(\tilde{t},\tilde{R}) = \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}^2}\right)\pi\left(\left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^{-2} t, \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^2 R + \tilde{r}\right). \tag{9}$$

In terms of the dimensionless quantities, the wealth process can be rewritten as

$$dX(t) = \pi(t)dt + \pi(t)dW(t), \tag{10}$$
$$X(0) = x.$$

Now we redefine $S(t)$ to be the discounted stock price, i.e., $e^{-\tilde{r}\tilde{t}}S(\tilde{t})$. Then the dynamic equation for dS expressed in terms of the dimensionless time t is

$$dS(t) = S(t)\,\alpha(dt + dW(t)),$$

where $\alpha \triangleq \sigma^2/(\tilde{\mu}-\tilde{r})$ is a dimensionless parameter. We comment that α will not appear in the rest of the paper since the following derivations are based only on Eq. (10) in which α is absorbed into the definition of $\pi(\cdot)$ (see Eq. (6)).

Obviously, the dimensionless investment horizon T can be expressed as $T = \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^2 \tilde{T}$, which depends not only on the dimensional investment horizon $\tilde{T}$, but also on the dimensional drift $\tilde{\mu}$, interest rate $\tilde{r}$ and volatility $\tilde{\sigma}$. Therefore, several factors can make T large, namely: long time horizon $\tilde{T}$, large drift $\tilde{\mu}$, and small volatility $\tilde{\sigma}$.

It will be shown later in the paper that when one follows the MMV, SPM or PUM strategies over a long investment horizon, there is a threshold in the dimensionless target return rate, above/below which the bankruptcy surely/never occurs. This threshold is $R^* = \frac{1}{2}$ for the MMV and SPM strategies and $R^* = 2$ for the PUM strategy. From Eq. (7), the threshold in undiscounted dimensional target return rate is:

$$\tilde{R}^* = \frac{1}{2}\left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^2 + \tilde{r} \tag{11}$$

for the MMV and SPM strategies, and

$$\tilde{R}^* = 2\left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^2 + \tilde{r} \tag{12}$$

for the PUM strategy.

We now provide an example illustrating the importance of the threshold for these three tamed strategies in terms of the dimensional target return rate $\tilde{R}$. Assuming the dimensional parameters are $\tilde{r} = 4\%$, $\tilde{\mu} = 10\%$ and $\tilde{\sigma} = 20\%$, then from Eq. (11) the threshold in undiscounted dimensional target return rate is 8.5% when one applies the MMV strategy or the SPM strategy over a long investment horizon. If an investor adopts the PUM strategy, then from Eq. (12) the threshold in undiscounted dimensional target return rate is 22%, which is much higher than that for the MMV strategy or the SPM strategy.

For the MMV, SPM and PUM strategies, the target return rate remains constant over the whole investment horizon. It is desirable to design a strategy such that the investor strategically and dynamically adjusts the target return rate. To reduce the probability of bankruptcy, the investor will lower the target when the portfolio wealth is low, and raise the target when the wealth is high. Although this is beyond the scope of the present paper, which is to study the bankruptcy probability of known strategies in the literature, it is an important practical issue. A study of portfolio selection based on the constraint on value-at-risk over the immediate time interval $[t, t+\tau]$ for any t before the investment horizon can be found in the work of Cuoco et al. (2001).

The outline of the paper is as follows: In Section 2 we analyze the long-term bankruptcy probability for the three tamed strategies and determine the threshold for each of them. In Section 3 we study the bankruptcy probability over a finite time horizon. In Section 4 we reveal the relation between the three tamed strategies and compare the strategies from different perspectives. In Section 5 we present the results in the multi-asset market. In Section 6 we present an empirical study based on the Dow Jones Industrial Average Index to confirm the results. Section 7 concludes. All derivations and proofs are given in Appendix A.

2. Bankruptcy Probability in long-term investments

In this section we survey the three tamed portfolio strategies. To describe the risk characteristics of a strategy, we introduce the following risk measure of bankruptcy probability up to the time horizon T: $Brp(T) \triangleq P(\inf_{0\leqslant t\leqslant T} X(t) \leqslant 0)$. For tamed strategies with wealth bounded below by zero, this definition of bankruptcy probability is equivalent to

$$Brp(T) = P(\inf_{0\leqslant t\leqslant T} X(t) = 0) = P(X(T) = 0), \tag{13}$$

which says that bankruptcy is an absorbing state (cf. Karatzas (1997), p.34). Because when the wealth becomes zero, any investment in the risky asset will lead to a negative wealth with a positive probability, which violates the nonnegative wealth restriction.

We will focus on the asymptotic behavior of bankruptcy probability, namely $\lim_{T\to\infty} Brp(T)$, which is clearly important for long-term investments. First, we derive analytical formulas for bankruptcy probabilities of the three tamed strategies under a finite investment horizon. Then, we show that in the long investment horizon limit ($T \to \infty$), there exists a unique threshold R^* in the target return rate for each strategy. When the target return rate is set below R^*, the bankruptcy will never occur. Otherwise, the portfolio will surely go bankrupt, i.e., the wealth will surely reach zero or become arbitrarily close to zero.

2.1. Modified Mean-Variance (MMV) Strategy

It is known that the optimal wealth process $X(t)$ in the continuous-time mean-variance portfolio selection setting can become negative during the investment period (see Eq. (A.10) for details). To overcome this problem, Bielecki et al. (2005) studied the mean-variance problem under the nonnegative wealth restriction. The formulation of this problem is:

$$\begin{aligned} \text{Goal: } &\min\ Var(X(T)) \\ &s.t.\ \ E(X(T)) = xe^{R_1 T}, \\ &\qquad X(t) \geqslant 0\ \ a.s.,\ \ \forall t \in [0,T] \end{aligned}$$

which has the solution

$$\pi_1(t) = \omega_1\,\Phi(-d_-(t,y(t))) - X(t), \tag{14}$$
$$X(t) = \omega_1\,\Phi(-d_-(t,y(t))) - \Phi(-d_+(t,y(t)))\,y(t), \tag{15}$$

where

$$y(t) = \omega_2\,e^T\,e^{-\frac{3}{2}t - W(t)}, \tag{16}$$
$$d_+(t,y) = \frac{\ln(y/\omega_1) + \frac{1}{2}(T-t)}{\sqrt{T-t}}, \tag{17}$$
$$d_-(t,y) = d_+(t,y) - \sqrt{T-t}. \tag{18}$$

Here $W(t)$ is the standard Brownian motion and (ω_1, ω_2) is the unique solution determined by the following equations:

$$\begin{cases} \omega_1\,\Phi\left(\frac{\ln(\omega_1/\omega_2)-\frac{1}{2}T}{\sqrt{T}}\right) - \omega_2\,e^T\,\Phi\left(\frac{\ln(\omega_1/\omega_2)-\frac{3}{2}T}{\sqrt{T}}\right) = x \\ \omega_1\,\Phi\left(\frac{\ln(\omega_1/\omega_2)+\frac{1}{2}T}{\sqrt{T}}\right) - \omega_2\,\Phi\left(\frac{\ln(\omega_1/\omega_2)-\frac{1}{2}T}{\sqrt{T}}\right) = x\,e^{R_1 T}, \end{cases} \tag{19}$$

where $\Phi(\cdot)$ denotes the cumulative normal distribution function. Notice that ω_1 and ω_2 are functions of T and R_1 (proportional dependence on x is trivial).

Eq. (15) for $X(T)$ can be simplified to

$$X(T) = \max(0, \omega_1 - \omega_2\, e^{-\frac{1}{2}T - W(T)}) \triangleq \left(\omega_1 - \omega_2\, e^{-\frac{1}{2}T - W(T)}\right)^+, \tag{20}$$

which together with Eq. (13) leads to

$$\begin{aligned} Brp(T) &= P\left(\omega_1 - \omega_2\, e^{-\frac{1}{2}T - W(T)} \leqslant 0\right) \\ &= 1 - \Phi\left(\frac{1}{\sqrt{T}}\left(\ln\left(\frac{\omega_1}{\omega_2}\right) + \frac{1}{2}T\right)\right). \end{aligned} \tag{21}$$

Because ω_1 and ω_2 depend on R_1, $Brp(T)$ has the following asymptotic behavior (see Appendix A for derivations)

$$\lim_{T\to\infty} Brp(T) = \begin{cases} 0, \ R_1 < \frac{1}{2} \\ \frac{1}{2}, \ R_1 = \frac{1}{2} \\ 1, \ R_1 > \frac{1}{2}. \end{cases} \tag{22}$$

Equation (22) shows that there exists a threshold $R_1^* = \frac{1}{2}$ when applying the long-term Modified Mean-Variance strategy. An investor should set his/her target return rate less than this threshold, which guarantees no bankruptcy. Otherwise, the investor will surely lose everything in a long-term investment.

2.2. Shortfall Probability Minimization (SPM) Strategy

This strategy is designed to minimize the shortfall probability of the portfolio value below a specified wealth level at a given investment horizon. This problem can be expressed as:

$$\text{Goal: } \min_{\pi} P\left[X(T) < xe^{R_2 T}\right].$$

Browne (1999, 2000) studied this problem and gave an explicit solution

$$\pi_2(t) = \frac{x\, e^{R_2 T}}{\sqrt{T-t}}\phi\left(\Phi^{-1}\left(\frac{X(t)}{x\, e^{R_2 T}}\right)\right), \tag{23}$$

$$X(t) = xe^{R_2 T}\Phi\left(\frac{1}{\sqrt{T-t}}\left[W(t) + t + \sqrt{T}\Phi^{-1}\left(e^{-R_2 T}\right)\right]\right) \tag{24}$$

for $(0 \leqslant t < T)$, where $\phi(\cdot)$ denotes the density function of a standard normal variable,

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}.$$

Unlike the MMV strategy where the target return rate R_1 is the same as the expected return rate, here the target return rate R_2 differs from the expected return rate, because

$$E(X(T)) = xe^{R_2 T}\,\Phi\left(\sqrt{T} + \Phi^{-1}\left(e^{-R_2 T}\right)\right) \leqslant xe^{R_2 T}. \tag{25}$$

However, it is easy to check that

$$\lim_{T\to\infty}\frac{1}{T}\ln\left(E(X(T)/x)\right)=R_2,\ \ \forall R_2<\frac{1}{2}, \tag{26}$$

which means that the target return rate R_2 is the same as the expected return rate in the long investment horizon limit provided that $R_2<\frac{1}{2}$.

Eq. (24) says that the final wealth $X(T)$ satisfies a binomial distribution, i.e.,

$$X(T)=\begin{cases} xe^{R_2T}, \text{ if } & W(T)+T+\sqrt{T}\Phi^{-1}\left(e^{-R_2T}\right)>0 \\ 0, \qquad \text{if } & W(T)+T+\sqrt{T}\Phi^{-1}\left(e^{-R_2T}\right)<0 \end{cases} \tag{27}$$

which leads to

$$\begin{aligned} Brp(T) &= P\left(W(T)+T+\sqrt{T}\Phi^{-1}\left(e^{-R_2T}\right)<0\right) \\ &= 1-\Phi\left(\sqrt{T}+\Phi^{-1}\left(e^{-R_2T}\right)\right). \end{aligned} \tag{28}$$

Using the asymptotic expansions of $\Phi^{-1}(\cdot)$ (see Bailey (1981)), we have the following expression in the limit $T\to\infty$

$$\Phi^{-1}\left(e^{-R_2T}\right)=-\sqrt{2R_2T}+\frac{\ln(4\pi R_2T)}{2\sqrt{2R_2T}}+O\left(\frac{1}{T}\right). \tag{29}$$

Substituting the expression (29) into Eq. (28), we obtain

$$\lim_{T\to\infty}Brp(T)=\begin{cases} 0, \ R_2<\frac{1}{2} \\ \frac{1}{2}, \ R_2=\frac{1}{2} \\ 1, \ R_2>\frac{1}{2}. \end{cases} \tag{30}$$

Equation (30) shows that the threshold in target return rate is $R_2^*=\frac{1}{2}$ for the SPM strategy.

2.3. Power Utility Maximization (PUM) Strategy

This is a classic optimal investment strategy in the continuous-time model introduced by Merton (1971). The formulation of this problem is

$$\text{Goal: }\max_{\pi} E\left[\frac{1}{\gamma}(X(T))^{\gamma}\right]$$

which has the solution

$$\pi_3(t)=\frac{1}{1-\gamma}X(t). \tag{31}$$

The relative risk aversion parameter γ of the power utility function is related to the target return rate R_3 by the following identity:

$$\gamma=1-\frac{1}{R_3}, \tag{32}$$

which is obtained from the equality

$$E(X(T)) = x\,\exp\left(\frac{1}{1-\gamma}T\right) \triangleq x\,\exp(R_3 T). \tag{33}$$

Notice that, by definition, the target return R_3 here is also the expected return rate of the wealth. In terms of R_3, $X(t)$ can be expressed as follows:

$$X(t) = x\exp\left(\left(R_3 - \frac{R_3^2}{2}\right)t + R_3\,W(t)\right). \tag{34}$$

For this strategy, the wealth is always positive, i.e., $X(t) > 0$, $\forall 0 \leqslant t \leqslant T$, which means zero probability of bankruptcy. However, one can ask the following question instead: Over a given time horizon what is the probability that the portfolio value falls below δ fraction of its initial wealth x, namely,

$$Brp(\delta, T) \triangleq P(\inf_{0\leqslant t\leqslant T} X(t) \leqslant \delta\,x).$$

Using the joint distribution of $(W(t), \inf_{0\leqslant s\leqslant t} W(s))$ and the Girsanov theorem, we can calculate this bankruptcy probability (derivation can be found in Appendix A):

$$Brp(\delta, T) = \Phi\left(\frac{\ln\delta - aT}{b\sqrt{T}}\right) + \Phi\left(\frac{\ln\delta + aT}{b\sqrt{T}}\right)e^{\frac{2a\ln\delta}{b^2}}, \tag{35}$$

where $a = R_3 - R_3^2/2$ and $b = R_3$.

The asymptotic behavior of $Brp(\delta, T)$ in the limit $T \to \infty$ is:

$$\lim_{T\to\infty} Brp(\delta, T) = \begin{cases} e^{(\frac{2}{R_3}-1)\ln\delta}, & \text{if } R_3 < 2 \\ 1, & \text{if } R_3 \geqslant 2. \end{cases} \tag{36}$$

Equation (36) shows that no matter how small the value of δ is, as long as the target return rate R_3 is sufficiently large and that the investment horizon is long enough, the value of the portfolio will surely reach $\delta\,x$. In particular, in the limit $\delta \to 0$ we have

$$\lim_{T\to\infty} Brp(0^+, T) = \begin{cases} 0, \ R_3 < 2 \\ 1, \ R_3 \geqslant 2. \end{cases} \tag{37}$$

So the PUM strategy also has a threshold of $R_3^* = 2$ for a long-term investment.

We now provide two ways to understand this threshold, one from the wealth growth rate perspective and the other from financial interpretation of the investment strategy. (i) It is easy to verify that $d\ln X(t) = R_3(1 - \frac{R_3}{2})dt + R_3 dW(t)$. So the log of wealth will be negative infinity as $t \to \infty$ if $R_3 \geqslant 2$. The corresponding threshold for the relative risk aversion parameter is $\gamma^* = 1 - \frac{1}{R_3^*} = \frac{1}{2}$ (cf. Eq. 32). Thus a long-term investor should only use the power utility function with $\gamma < \frac{1}{2}$, which is pointed out in Chapter 2 of Yang (2006). (ii) When $R < 1$, one keeps $(1 - R)X > 0$ amount of his/her wealth in the money market. Obviously, bankruptcy will never occur in this case. When $R > 1$, one needs to borrow $(R-1)X$ from the money market account. In the case of $R < 2$, even if the investor loses all borrowed money $(R-1)X$, he/she still has a net asset value $X' = (2-R)X$ after paying the debt, which allows the investor to continue the investment strategy by borrowing $(R-1)X'$. However, when $R > 2$, bankruptcy occurs if the investor loses all the borrowed money.

We have proven that in the long investment horizon limit, the bankruptcy probability tends to one when the target return rate is set above the threshold. One may wonder, in such a case, how the condition $E(X(T)) = xe^{RT}$ on the final wealth can be satisfied. We explain this paradox in Appendix B.

3. Bankruptcy Probability in Finite Time Horizon

In this section we discuss and compare the bankruptcy probabilities of the MMV, the SPM and the PUM strategies over a finite time horizon, which are given analytically as functions of R and T by Eqs. (21), (28) and (35), respectively.

Figure 1 presents the bankruptcy probabilities as a function of R for T =1, 5 and 25 for the MMV, the SPM and the PUM strategies. For the PUM strategy we set δ in Eq. (35) to 0.001, i.e., losing 99.9% of the initial wealth. Obviously, for each strategy the shape of the curve converges to a step function as $T \to \infty$, and the jump point of the step function is located at the threshold value R^* as it should be.

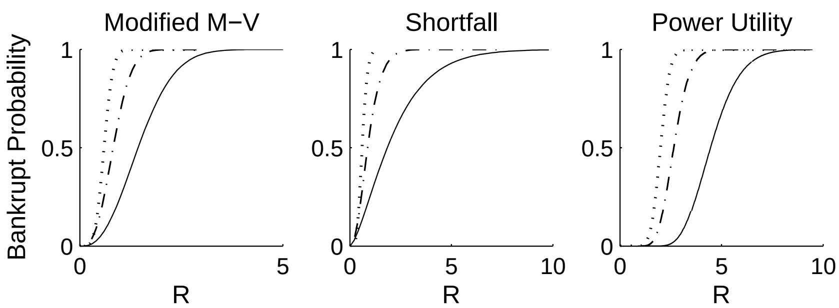

Figure 1 Bankruptcy probability as a function of R for different T. Solid, dash-dotted, dotted lines are results for $T = 1$, 5 and 25, respectively

In Figure 2 we compare the bankruptcy probabilities among the MMV, the SPM and the PUM strategies over same time horizon. The values of the parameters are the same as those in Figure 1. As we mentioned earlier, for the MMV and the PUM strategies, the target return rate is the same as the expected return rate, i.e., $R = R_e$. However, for the SPM strategy, although $R_2 = R_e$ holds in the limit as $T \to \infty$ (see Eq. (26)), they are different in a finite time horizon (see Eq. (25)). Therefore, one needs to use R_e instead of R_2 in the SPM strategy for comparing different strategies in finite T. R_e is related to R_2 by the following equation:

$$E(X(T)) = xe^{R_2 T}\,\Phi\left(\sqrt{T} + \Phi^{-1}\left(e^{-R_2 T}\right)\right) = xe^{R_e T},$$
$$\text{i.e., } R_e = R_2 + \frac{1}{T}\ln\Phi\left(\sqrt{T} + \Phi^{-1}\left(e^{-R_2 T}\right)\right).$$

Figure 2 shows that under the same expected return rate R_e, the PUM strategy has the lowest bankruptcy probability among all three strategies, and the SPM strategy has the highest bankruptcy probability. We will further discuss the relation among these three seemingly different strategies in Section 4.

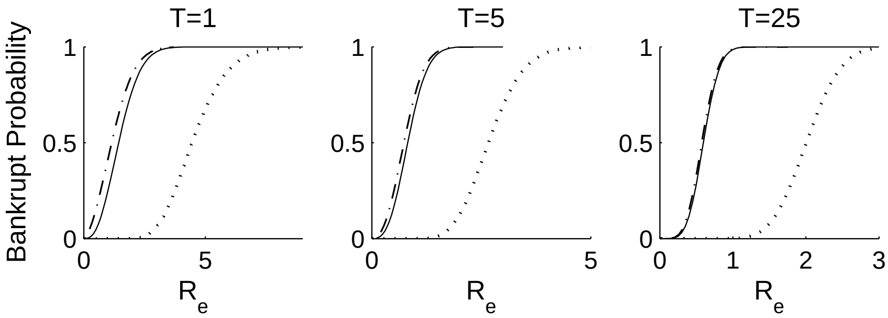

Figure 2 Bankruptcy probability as a function of R for different values of T. The solid, dash-dotted, dotted lines are results for the MMV, the SPM and the PUM strategies, respectively.

Figure 3 shows the bankruptcy probabilities of the MMV, SPM and PUM strategies as a function of investment time horizon T for different values of R_e/R^*. The situation for $R_e \geq R^*$ is plotted in the top row, whereas the case $R_e < R^*$ is plotted in the bottom row, because the vertical scales of the top and bottom rows are very different. It is clear that for the PUM strategy, the bankruptcy probability is an increasing function of T regardless of R_e (cf. Eq. (40)). However, for the MMV and SPM strategies, the bankruptcy probabilities are increasing functions of T only when $R_e \geq R^*$; there exists a maximum bankruptcy probability P^* located at $T = T^*$ when $R_e < R^*$. Obviously, one should avoid choosing an investment horizon near T^* when applying these two strategies.

Now, let us determine T^* which is governed by the equation $\frac{\partial Brp(T)}{\partial T}|_{T=T^*} = 0$. It is straightforward to derive the following results. For the MMV strategy, T_1^* satisfies:

$$\frac{\partial Brp(T_1^*)}{\partial T_1^*} = -\phi\left(\frac{1}{\sqrt{T_1^*}}\left(\ln\eta + \frac{1}{2}T_1^*\right)\right)\left(-\frac{1}{2T_1^{*\frac{3}{2}}}\ln\eta + \frac{1}{\eta\sqrt{T_1^*}}\frac{d\eta}{dT_1^*} + \frac{1}{4\sqrt{T_1^*}}\right) = 0,$$

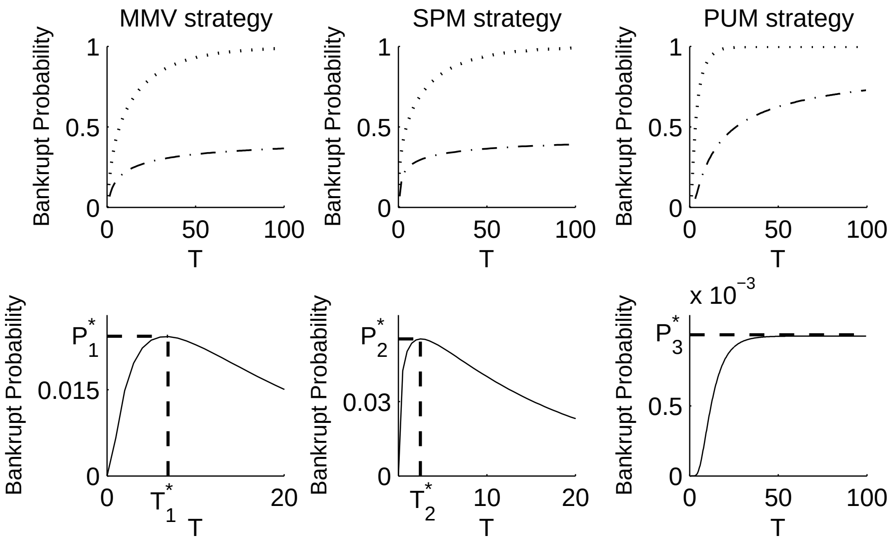

Figure 3 Bankruptcy probability as a function of T for different values of R_e. Solid, dash-dotted, dotted lines stand for $R_e = 0.5R^*, R^*$ and $1.5R^*$, respectively.

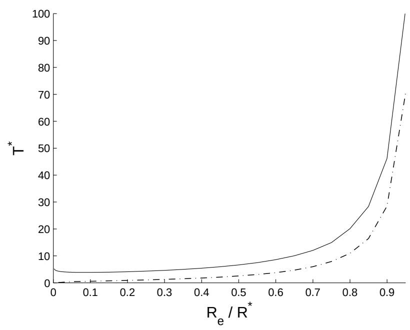

Figure 4 Relation between T^* and R_e/R^*. Solid and dash-dotted lines are results for the MMV and the SPM strategy, respectively.

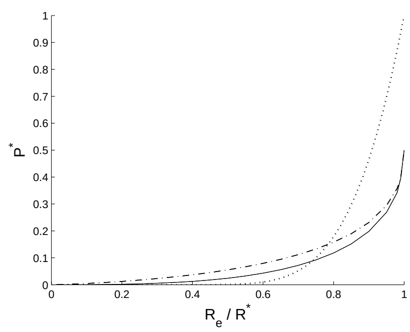

Figure 5 Relation between P^* and R_e/R^*. Solid, dash-dotted and dotted lines are results for the MMV, the SPM and the PUM strategies, respectively.

namely,

$$-\frac{1}{2T_1^*}\ln\eta+\frac{1}{\eta}\frac{d\eta}{dT_1^*}+\frac{1}{4}=0 \tag{38}$$

where $\eta=\omega_1/\omega_2$ is a function of T_1^*, and satisfies Eq. (A.2) (see Appendix A for details). For the SPM strategy, T_2^* satisfies:

$$\frac{\partial Brp(T_2^*)}{\partial T_2^*}=-\phi\left(\sqrt{T_2^*}+\Phi^{-1}\left(e^{-R_2T_2^*}\right)\right)\left(\frac{1}{2\sqrt{T_2^*}}-\frac{R_2e^{-R_2T_2^*}}{\phi(\Phi^{-1}(e^{-R_2T_2^*}))}\right)=0,$$

namely,

$$\frac{1}{2\sqrt{T_2^*}}-\frac{R_2e^{-R_2T_2^*}}{\phi(\Phi^{-1}(e^{-R_2T_2^*}))}=0. \tag{39}$$

For the PUM strategy:

$$\frac{\partial Brp(T)}{\partial T}=-\frac{\ln\delta}{R_3T^{\frac{3}{2}}}\phi\left(\frac{\ln\delta-(R_3-R_3^2/2)T}{R_3\sqrt{T}}\right)>0. \tag{40}$$

$Brp(T)$ is an increasing function of T because $\ln\delta<0$, hence $T_3^*=\infty$. The maximum bankruptcy probability $P=Brp(T^*)$ for the MMV, SPM and PUM strategies can be easily determined from Eqs. (21), (28) and (35), respectively.

As R_e/R^* approaches one from below, T^* for the MMV and the SPM strategies approaches infinity; see Figure 4. Notice that T^* for the PUM strategy is always infinite. Furthermore, under the limit $R_e/R^*\to1$, the maximum bankruptcy probability P^* for the MMV and the SPM strategies approaches $\frac{1}{2}$ and that for the PUM strategy approaches 1; see Figure 5. This is consistent with our earlier conclusions about long-term investments under these three strategies; see Eqs. (22), (30) and (37) for details. Obviously, sure bankruptcy for long-term investments will occur when $R_e/R^*>1$.

On the other hand, as R_e approaches zero, three curves in the bottom row of Figure 3 will all collapse to the x-axis, i.e., bankruptcy probabilities become zero, which is clearly demonstrated in Figure 5, as $P^*\to0$ when $R_e\to0$ for all three strategies. This can be easily proven by letting $R\to0$ in Eqs. (21), (28) and (35). For the MMV strategy, there is an interior maximum point T^* whose value never drops below 3.8462. For the SPM strategy, T^* approaches zero as $R_e\to0$.

4. Comparison among Strategies

In previous sections, we have studied the bankruptcy probabilities of the MMV, SPM and PUM strategies. In this section, we compare these strategies and reveal that some do behave the same in certain parameter ranges. Such a comparison study can be carried out from two different perspectives: the allocation of wealth to risky asset $\pi(t)$ and utility function maximization.

4.1. Asset allocation perspective

The following theorem shows that when T is small, the MMV strategy and the PUM strategy allocate the same amount of money to the risky asset, i.e., $\pi_1(t) \sim \pi_3(t)$. When T is very large, the MMV strategy and the SPM strategy allocate the same amount of money to the risky asset, i.e., $\pi_1(t) \sim \pi_2(t)$. Here the symbol $\sim$ denotes the equivalence relation.

THEOREM 1. *For a given finite target return rate R, we have*

$$\lim_{T\to 0} \pi_1(t) = \lim_{T\to 0} \pi_3(t), \quad \lim_{T\to\infty} \pi_1(t) = \lim_{T\to\infty} \pi_2(t) \quad (0 \leqslant t \leqslant T).$$

Theorem 1 explains why the MMV and the SPM strategies have the same threshold in target return rate for a long-term investment.

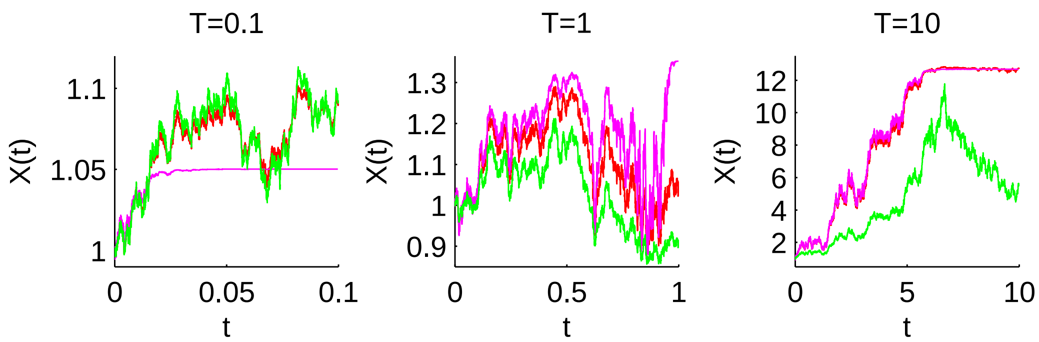

Figure 6 Sample paths of the MMV, SPM and PUM strategies for different time horizons. Red, purple and green lines stand for the MMV, the SPM and the PUM strategies, respectively.

For an intuitive demonstration, we plot in Figure 6 some sample paths of the three strategies. Here we set $R = \frac{1}{4}$, which is half of the threshold of the MMV and the SPM strategies, and one eighth of the threshold of the PUM strategy. The stock path is generated randomly by a Monte Carlo simulation. Based on the same stock path, we obtain the corresponding wealth paths for different portfolio strategies from Eqs. (15), (24) and (34). As we move from the left panel to the right panel, T increases. These panels show that the path of the MMV strategy is close to that of the PUM/SPM strategy when T is small/large (see first/third panel). These are visual demonstrations of Theorem 1. We comment that the purple curves for the SPM strategy become flat at a later portion of the horizon. This is due to the fact that the strategy minimizes the probability of the value of portfolio below a given level. Once that level is reached, the strategy will no longer allocate any wealth to the risky asset.

To explain the importance of the threshold in target return rate in a long-term investment, we plot in Figure 7 the wealth paths of these strategies under two different target return rates:

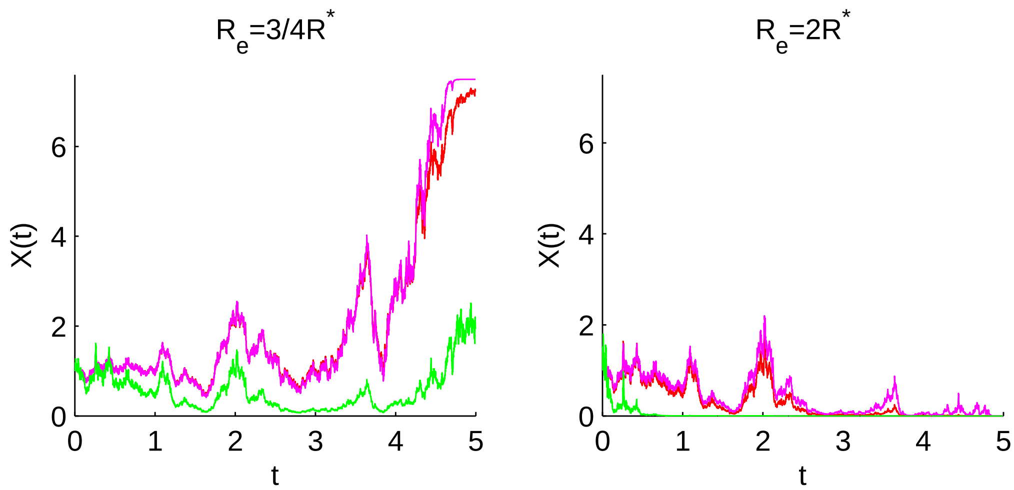

Figure 7 Comparison of the MMV, SPM and PUM strategies for different target return rates under the same stock path. Red, purple and green lines stand for the MMV, SPM and PUM strategies, respectively.

one below the threshold and the other above. Figure 7 shows that strategies with target return rates below the threshold perform better than those with the target return rates above the threshold. It means that the investors may increase their target return rate as long as it is below the threshold. Setting the target return rate above the threshold will degrade, rather than improve, the performance of the portfolio.

4.2. Utility functions maximization perspective

The utility maximization method is a useful and popular approach in portfolio selection. The three tamed strategies that we focus on in this paper can be treated under the same framework of utility maximization, namely maximizing $E(U(X_T))$ with different utility functions $U(\cdot)$. We will compare the three strategies from utility function maximization perspective in this section.

The utility function corresponding to the MMV strategy is given by Xia (2005),

$$U(y) = -((\lambda - y)^+)^2, \qquad \lambda \geqslant x, \tag{41}$$

which can be written in the following form after an equivalent linear transformation

$$U_1(y) = -((1 - \frac{y}{\lambda})^+)^2 + 1, \qquad \lambda \geqslant x. \tag{42}$$

where the parameter λ is related to the target return rate R_1 by the expression (derivation can be found in Appendix A)

$$\lambda = x\,\frac{e^{(R_1+1)T} - 1}{e^T - 1}. \tag{43}$$

The utility function for the SPM strategy is:

$$U_2(y) = 1_{\{y \geqslant xe^{R_2 T}\}}. \tag{44}$$

It is a direct result of the following equation:

$$\min_{\pi} P\left[X(T) < xe^{R_2 T}\right] = \max_{\pi} E\left[1_{\{X(T) \geqslant xe^{R_2 T}\}}\right],$$

which is convex for the part below the target and concave for the part above the target. This type of utility function is described in Benartzi and Thaler (1999) and Berkelaar et al. (2004).

Obviously, the utility function for the PUM strategy is equivalent to:

$$U_3(y) = \frac{1}{\gamma} y^{\gamma} + 1, \tag{45}$$

where the risk aversion parameter γ and the target return rate R_3 are related by Eq. (32).

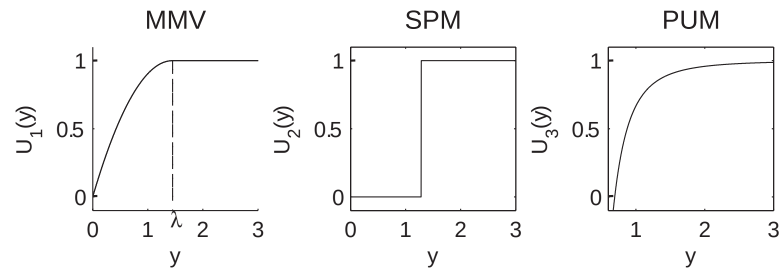

Figure 8 Comparison of the corresponding utility functions of the MMV, the SPM and the PUM strategies.

The three utility functions are parameterized by the target return rate R. In addition, the utility functions corresponding to the MMV and the SPM strategies depend on the investment horizon T as well. The curves in Figure 8 correspond to the utility functions for the MMV, the SPM and the PUM strategies given by Eqs. (42), (44) and (45), where the parameters are $R = \frac{1}{4}$ and $T = 1$. We comment that the shape of the utility function of the MMV strategy looks like a combination of different portions of utility functions of the SPM and the PUM strategies: the segment $y < \lambda$ behaves like the PUM strategy and the segment $y \geqslant \lambda$ behaves like the SPM strategy.

We now quantify these observations. Let us first examine the limit $T \to 0$, under which the wealth distribution for $X(T)$ is almost like a delta function centered at the initial wealth x. Therefore, only the local behavior of the utility function around x is relevant. The local behavior of a utility function can be uniquely characterized by the relative risk aversion (RRA) function:

$$RRA(y) = -\frac{yU''(y)}{U'(y)}.$$

In order to determine the value of the relative risk aversion function for the MMV strategy at x, we must know whether x is greater or less than λ. From Eq. (43), we have

$$\lim_{T\to 0}\lambda = \lim_{T\to 0}\frac{e^{(R_1+1)T}-1}{e^T-1} = x\,(R_1+1) > x.$$

After some algebraic manipulations, we obtain the RRA for the MMV strategy

$$RRA_1(x) = \frac{x}{\lambda - x} = \frac{1}{R_1}. \tag{46}$$

It is well known that the relative risk aversion function for the power utility is a constant

$$RRA_3(y) = 1-\gamma = \frac{1}{R_3}, \tag{47}$$

where the last equal sign of Eq. (47) comes from the relation in Eq. (32). Therefore, for a given target return rate R, the MMV and the PUM strategies are the same in the limit $T \to 0$.

We now examine the limit $T\to\infty$, under which the MMV and the SPM strategies are the same. Notice that the utility function of the MMV strategy becomes flat when $y > \lambda$, which shares the same character as the SPM strategy when $y > xe^{R_2 T}$. This means that an investor is indifferent to the unit increase of wealth when the wealth level is above a certain threshold. By setting $R = R_1 = R_2$, we can obtain that

$$\lim_{T\to\infty}\frac{\lambda}{e^{R_2 T}} = 1\,.$$

This result shows that the saturated wealth levels for the MMV and the SPM strategies are the same as $T\to\infty$. Furthermore, we can obtain the following theorem:

THEOREM 2. *For a given target return rate R, we have*

$$\lim_{T\to\infty} E(U_1(X_1(T))) = \lim_{T\to\infty} E(U_2(X_2(T))) = \begin{cases} 1, & R<\frac{1}{2} \\ \frac{1}{2}, & R=\frac{1}{2} \\ 0, & R>\frac{1}{2}. \end{cases} \tag{48}$$

Theorem 2 verifies the equivalence between the utility function of MMV and SPM strategies in the long investment horizon limit, and shows us again from the utility perspective why we should set the target return rate below the threshold for a long-term investment.

5. Strategies in multi-risk-asset market

As we stated in the introduction, our purpose is to study various strategies in a financial market which consists of n risky assets and a money market account. In this section, we show that all our conclusions in previous sections still hold in this multi-risk-asset market. All quantities in this section are dimensional ones; for brevity, we have drop the "˜" symbol associated with the dimensional quantities.

The price processes of the stocks satisfy the stochastic differential equation (SDE) given by Eq. (1). Let $\boldsymbol{\pi_t} = (\pi_t^{(1)}, ..., \pi_t^{(n)})'$ denote a vector control process in which $\pi_t^{(i)}$ describes the amount of money invested in the ith stock at time t. Then the discounted wealth process can be written as:

$$dX(t) = \sum_{i=1}^{n} \pi_t^{(i)}(\mu_i - r)dt + \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_t^{(i)} \sigma_{ij}\, dW_t^{(j)}. \tag{49}$$

We assume that the matrix $\boldsymbol{\sigma} = (\sigma)_{ij}$ is invertible. After introducing (column) vectors $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)'$, $\mathbf{1} = (1, ..., 1)'$, and

$$\boldsymbol{B} \triangleq \boldsymbol{\mu} - r\mathbf{1}, \tag{50}$$

$$\boldsymbol{\theta} \triangleq \boldsymbol{\sigma^{-1}}\,\boldsymbol{B}, \tag{51}$$

we can rewrite Eq. (49) as

$$\begin{aligned} dX(t) &= \boldsymbol{\pi_t'}(\boldsymbol{\mu} - r\mathbf{1})dt + \boldsymbol{\pi_t'}\boldsymbol{\sigma}\, d\boldsymbol{W_t}, \\ &= \boldsymbol{\pi_t'}\boldsymbol{\sigma}\,\boldsymbol{\theta}\, dt + \boldsymbol{\pi_t'}\boldsymbol{\sigma}\, d\boldsymbol{W_t}. \end{aligned} \tag{52}$$

The three tamed strategies in the multi-risk-asset market can be expressed in terms of these vectors.

1. Modified Mean-Variance Strategy (Bielecki et al. (2005)):

$$\begin{aligned} \boldsymbol{\pi_1(t)} &= (\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\left[\omega_1\,\Phi(-d_-(t, y(t))) - X(t)\right], \\ X(t) &= \omega_1\,\Phi(-d_-(t, y(t))) - \Phi(-d_+(t, y(t)))\, y(t) \end{aligned} \tag{53}$$

where

$$\begin{aligned} y(t) &= \omega_2\, e^{|\boldsymbol{\theta}|^2 T}\, e^{-\frac{3}{2}|\boldsymbol{\theta}|^2 t - \boldsymbol{\theta'}\boldsymbol{W_t}}, \\ d_+(t, y) &= \frac{\ln(y/\omega_1) + \frac{1}{2}|\boldsymbol{\theta}|^2(T-t)}{|\boldsymbol{\theta}|\sqrt{T-t}}, \\ d_-(t, y) &= d_+(t, y) - |\boldsymbol{\theta}|\sqrt{T-t} \end{aligned}$$

and (ω_1, ω_2) is the unique solution to the following system of equations:

$$\begin{cases} \omega_1\,\Phi\left(\frac{\ln(\omega_1/\omega_2) - \frac{1}{2}|\boldsymbol{\theta}|^2 T}{|\boldsymbol{\theta}|\sqrt{T}}\right) - \omega_2\, e^{|\boldsymbol{\theta}|^2 T}\,\Phi\left(\frac{\ln(\omega_1/\omega_2) - \frac{3}{2}|\boldsymbol{\theta}|^2 T}{|\boldsymbol{\theta}|\sqrt{T}}\right) = x, \\ \omega_1\,\Phi\left(\frac{\ln(\omega_1/\omega_2) + \frac{1}{2}|\boldsymbol{\theta}|^2 T}{|\boldsymbol{\theta}|\sqrt{T}}\right) - \omega_2\,\Phi\left(\frac{\ln(\omega_1/\omega_2) - \frac{1}{2}|\boldsymbol{\theta}|^2 T}{|\boldsymbol{\theta}|\sqrt{T}}\right) = x\, e^{R_1 T}. \end{cases}$$

2. Shortfall Probability Minimization Strategy (Browne (1999)):

$$\begin{aligned} \boldsymbol{\pi_2(t)} &= x\, e^{R_2 T}\,\frac{(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}}{|\boldsymbol{\theta}|\sqrt{T-t}}\,\phi\left(\Phi^{-1}\left(\frac{X(t)}{x\, e^{R_2 T}}\right)\right), \\ X(t) &= x\, e^{R_2 T}\,\Phi\left(\frac{1}{|\boldsymbol{\theta}|\sqrt{T-t}}\left[\boldsymbol{\theta'}\,\boldsymbol{W_t} + |\boldsymbol{\theta}|^2 t + |\boldsymbol{\theta}|\sqrt{T}\Phi^{-1}\left(e^{-R_2 T}\right)\right]\right). \end{aligned} \tag{54}$$

3. Power Utility Maximization Strategy (Merton (1971)):

$$\begin{aligned}\boldsymbol{\pi_3(t)} &= \frac{(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}}{1-\gamma}X(t), \\ X(t) &= x\exp\left(\left(\frac{|\boldsymbol{\theta}|^2}{1-\gamma}-\frac{1}{2}\frac{|\boldsymbol{\theta}|^2}{(1-\gamma)^2}\right)t+\frac{\boldsymbol{\theta' W_t}}{1-\gamma}\right).\end{aligned} \tag{55}$$

The target return rate R_3 is related to γ as follows:

$$E(X(T)/x) = \exp\left(\frac{|\boldsymbol{\theta}|^2}{1-\gamma}T\right) \triangleq \exp(R_3 T),$$

from which we obtain $\gamma = 1 - \frac{|\boldsymbol{\theta}|^2}{R_3}$.

Notice that, from Eqs. (53), (54) and (55), the allocation of wealth to each risky asset is proportional to the vector $(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}$. We further comment that this vector is independent of strategies and independent of an investor's preferences on target return rate R. With this important property, we can transform the multi-dimensional problem into a one-dimensional problem.

The total amount invested in the risky assets is $\boldsymbol{1'}\,\boldsymbol{\pi(t)}$, the fraction of which allocated to different risky assets is distributed according to the vector

$$\boldsymbol{u} \triangleq \frac{(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}}{\boldsymbol{1'}\left[(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\right]}.$$

Define μ_m and σ_m as follows:

$$\begin{aligned}\mu_m &\triangleq \boldsymbol{B'}\,\boldsymbol{u} = \frac{\boldsymbol{B'}\,(\boldsymbol{\sigma\sigma'})^{-1}\,\boldsymbol{B}}{\boldsymbol{1'}\left[(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\right]} = \frac{\boldsymbol{\theta'}\,\boldsymbol{\theta}}{\boldsymbol{1'}\left[(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\right]} = \frac{|\boldsymbol{\theta}|^2}{\boldsymbol{1'}\left[(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\right]}, \\ \sigma_m &\triangleq |\,\boldsymbol{\sigma'}\,\boldsymbol{u}| = \frac{|\boldsymbol{\sigma'}\,(\boldsymbol{\sigma\sigma'})^{-1}\,\boldsymbol{B}|}{\boldsymbol{1'}\left[(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\right]} = \frac{|\boldsymbol{\theta}|}{\boldsymbol{1'}\left[(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\right]}.\end{aligned}$$

Then, in terms of μ_m and σ_m, Eq. (52) for the wealth $X(t)$ can be rewritten as

$$dX(t) = \mu_m \pi(t)dt + \sigma_m \pi(t)dW(t),$$

where the corresponding $\pi(t)$ for the MMV, the SPM and the PUM strategies are

$$\begin{aligned}\pi_1(t) &= \boldsymbol{1'}\left[(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\right]\left[\omega_1\,\Phi(-d_-(t,y(t))) - X(t)\right], \\ \pi_2(t) &= x\,e^{R_2 T}\,\frac{\boldsymbol{1'}\left[(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\right]}{|\boldsymbol{\theta}|\sqrt{T-t}}\phi\left(\Phi^{-1}\left(\frac{X(t)}{x\,e^{R_2 T}}\right)\right), \\ \pi_3(t) &= \frac{\boldsymbol{1'}\,(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}}{1-\gamma}X(t),\end{aligned}$$

respectively. Note that $\boldsymbol{1'}\left[(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}\right]$ is a constant scalar in the expressions above.

Therefore, the n risky assets in our real market are equivalent to a single risk asset in a virtual market. This fictitious risky asset follows a geometric Brownian motion with drift μ_m and volatility σ_m. So we have transformed the multi-risk-asset problem to a single-risky-asset problem which we have discussed in detail in previous sections. The threshold in target return rate in the multi-risky-asset problem is

1. The MMV strategy: $R_1^* = \frac{1}{2}|\boldsymbol{\theta}|^2$,
2. The SPM strategy: $R_2^* = \frac{1}{2}|\boldsymbol{\theta}|^2$,
3. The PUM strategy: $R_3^* = 2|\boldsymbol{\theta}|^2$.

The corresponding version of Theorem 1 in multi-risky-asset market can be expressed as follows:

Theorem 3. *For a given target return rate R, we have*

$$\lim_{|\boldsymbol{\theta}|^2 T\to 0} \boldsymbol{\pi_1}(t) = \lim_{|\boldsymbol{\theta}|^2 T\to 0} \boldsymbol{\pi_3}(t), \quad \lim_{|\boldsymbol{\theta}|^2 T\to \infty} \boldsymbol{\pi_1}(t) = \lim_{|\boldsymbol{\theta}|^2 T\to \infty} \boldsymbol{\pi_2}(t), \quad (0 \leqslant t \leqslant T).$$

It should be pointed out that the vector $(\boldsymbol{\sigma\sigma'})^{-1}\boldsymbol{B}$ is the market portfolio in the capital asset pricing model, and plays a fundamental role in the theory of finance. The well-known two-fund separation theorem states that all investors will hold a combination of the market portfolio and the risk-free asset; an investor's risk preference only affects the allocation between the risk-free asset and the market portfolio, but not the relative weights among different risky assets. We see that this conclusion also holds for all three tamed continuous-time strategies discussed in the paper. This result is not a surprise when risky assets are log-normally distributed, and that the strategy is associated with a non-decreasing utility function maximization problem of the final wealth (see Section 4.2 for the utility functions associated with the MMV, the SPM and the PUM strategies), because they are the two necessary conditions for the two-fund separation theorem in continuous-time, which has been proven by Khanna and Kulldorff (1999).

6. Empirical Study

In this section, we use the "Dow Jones Industrial Average Index" (DJIAI) for our empirical studies and discussions. We will present two similar studies with different investment horizons.

We apply the MMV, SPM and PUM strategies for a ten-year investment over the period June 25, 1996 - June 28, 2006. We examine the case when the expected return rate has been set above the threshold value and the case below the threshold. We use the historical data of the DJIAI in the 10 years prior to the period in question, e.g., the period June 30, 1986 - June 24, 1996, to estimate the drift and volatility by the standard approach described in Chapter 12 of Hull (2003).

To conduct our empirical study, we use DJIAI index daily closed data, $S(t)$. We determine our allocation of wealth on the risky asset $\pi(t)$ based on Eqs. (14), (23) and (31) for the MMV, SPM and PUM strategies, respectively. Then, the corresponding wealth $X(t)$ is given by

$$X(t+\Delta t) = \pi(t)\frac{S(t+\Delta t)}{S(t)} + (X(t) - \pi(t))e^{\tilde{r}\Delta t}. \tag{56}$$

Here $\Delta t = 1/N$ where N is the number of trading days in a year. Since $X(t)$ is proportional to $X(0)$, we present all empirical results for $X(t)/X(0)$.

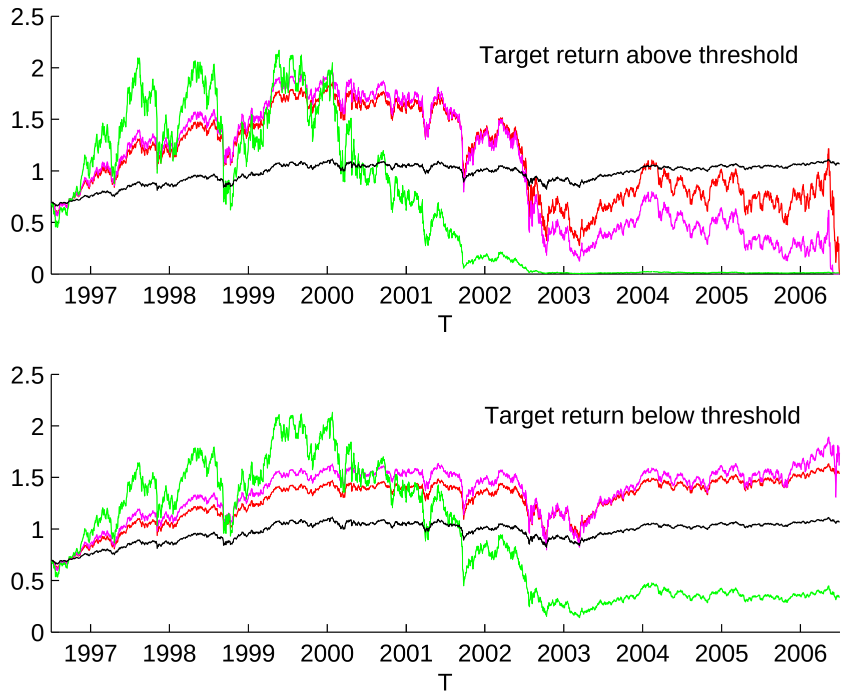

Figure 9 Comparison of the MMV, SPM and PUM strategies based on the Dow Jones Industrial Average Index over 10 years starting from June 1996. The portfolio is updated daily with $\tilde{r} = 3\%$, $\tilde{\mu} = 12.88\%$ and $\tilde{\sigma} = 17.06\%$. Red, purple, green and black lines stand for the MMV, SPM, PUM strategies and the DJIAI respectively.

The risk-free rate is set at $\tilde{r} = 3\%$ per annum, and the estimated model parameters are $\tilde{\mu} = 12.88\%$, $\tilde{\sigma} = 17.06\%$. Notice that the wealth in this paper is discounted, using Eq. (7). The threshold for the undiscounted dimensional target return rate for MMV and SPM strategy is $\tilde{R}^* = \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^2 R^* + \tilde{r} = 19.77\%$. We set $\tilde{R} = 24\%$ and $\tilde{R} = 16\%$, which correspond to the cases of the target return rate being above and below the threshold.

There are two reasons we set the target return rate 4% away from the threshold. First, due to the estimated error in calibrating the historical data, the estimated model parameters may not be as accurate as we want. Setting $\pm 4\%$ away from 19.77% increases the chances that these two target return rates locate on two sides of the threshold. Second, the time horizon is not long enough, e.g., in our empirical case, $T = \left(\frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}}\right)^2 \tilde{T} = 3.35 \ll \infty$. Therefore there is a range $(\tilde{R}^l, \tilde{R}^h)$ around $\tilde{R}^*$ in which bankruptcy probability changes rapidly from low to high (see Figure 1 for details). It is obvious that the longer the dimensionless horizon T is, the narrower the range $(\tilde{R}^l, \tilde{R}^h)$ will be. As $T \to \infty$, this range will collapse on to a single point $\tilde{R}^*$, which is the critical point of target return rate.

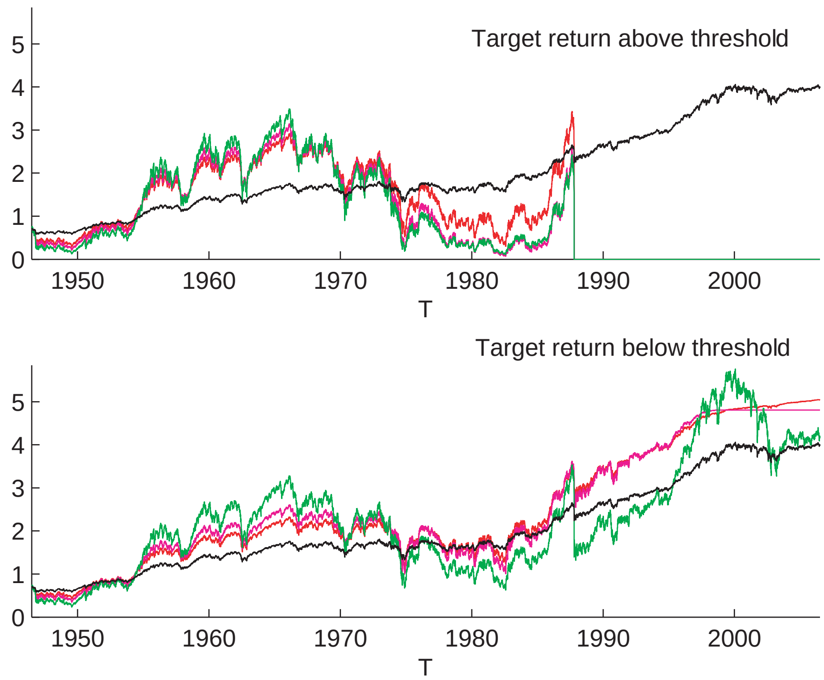

Figure 10 Comparison of the MMV, SPM and PUM strategies based on the Dow Jones Industrial Average Index over 60 years starting from June 1946. The portfolio is updated daily with $\tilde{r} = 3\%$, $\tilde{\mu} = 7.76\%$ and $\tilde{\sigma} = 14.31\%$. Red, purple, green and black lines stand for the MMV, SPM, PUM strategies and the DJIAI, respectively.

Due to the large fluctuation of the DJIAI, we adopt the log scale to plot the performance. The top panel of Figure 9 shows that the target return rate $\tilde{R} = 24\%$ leads to bankruptcy at the end of the investment horizon under the MMV and SPM strategies. The investment based on the MMV strategy bankrupted on June 28, 2006, and the investment based on the SPM strategy bankrupted on June 5, 2006. Note that both investments bankrupted on the end of investment horizon. This is due to the fact that near the end of the investment horizon, if the portfolio has not met the target return rate, it will allocate more wealth to the risky asset, which dramatically increases the risk of bankruptcy. When $\tilde{R} = 16\%$ which is below the threshold, both the MMV and SPM strategies perform much better than the DJIAI.

The threshold of undiscounted dimensional target return rate of the PUM strategy is much higher than that of the MMV and SPM strategies, e.g., 70.08% in this case. For comparison purpose, we also plot the portfolio process of the PUM strategy when the target return rate is set higher and lower than the threshold, i.e., 75% and 50%, respectively. In fact, we observe that when the target return rate is set to 75%, applying the PUM strategy will lead to a large loss, e.g., losing

99% of initial wealth on May 7, 2002. We comment that both the MMV and SPM strategies lead to bankruptcy at the 24% target return rate, while the bankruptcy does not occur for the PUM strategy even at the 50% target return rate. Therefore an investor should prefer the PUM strategy than the other two strategies. This is consistent with our theoretical finding presented in Figs. 1 and 2, namely the PUM strategy is much safer than the MMV and SPM strategies under the same target return rate.

To consider the case of extremely long-term investments, we calibrate the model parameters of the DJIAI over the past sixty years, e.g., June 25, 1946 - June 28, 2006. We find $\tilde{\mu} = 7.76\%$, $\tilde{\sigma} = 14.31\%$ which corresponds to the threshold $\tilde{R}^* = 8.54\%$ for the MMV and SPM strategies, the threshold $\tilde{R}_3^* = 25.1\%$ for the PUM strategy. Notice that these thresholds are much lower than the shorter horizon study. In the top panel of Figure 10, we set the target return rates to $\tilde{R} = 12\%$ for the MMV and SPM strategies, and $\tilde{R}_3 = 26\%$ for the PUM strategy, which are all higher than their respective long-term bankruptcy thresholds. All three strategies went bankrupt on the same day: October 19, 1987, due to the stock market crash on that day. In the bottom panel of Figure 10, the target return rates are $\tilde{R} = 8\%$ for the MMV and SPM strategies, and $\tilde{R}_3 = 20\%$ for the PUM strategy, which are all lower than their respective long-term bankruptcy thresholds. Notice that in this case none of the strategy went bankrupt even with the 1987 market crash. These results underscore how important the bankruptcy threshold is in setting the target return rate. General conclusions of the previous empirical study also apply here.

We comment that the dimensional threshold depends on $\tilde{\sigma}$ and $\tilde{\mu} - \tilde{r}$ (come from Eqs. (11) and (12)). If $\tilde{\sigma}$ is increased by a factor of 2, or $\tilde{\mu} - \tilde{r}$ is decreased by a factor of 2, then the dimensional threshold of the discounted wealth will almost be reduced by a factor of 4. Since there are many volatile or low growth stocks in the market, their long-term bankruptcy thresholds will be lower than the ones of the DJIAI. Therefore one must be careful when setting target return rates for these stocks.

7. Conclusion

Continuous-time portfolio selection strategies can be divided into two classes depending on whether the strategy imposes a lower bound on the wealth. Applying strategies without lower wealth bounds (untamed) leads to a sure bankruptcy in a long-term investment. The main conclusion of this paper is that for the three (MMV, SPM and PUM) tamed strategies with lower wealth bound of zero, there is a unique threshold in the target return rate above which sure bankruptcy will result in the long investment horizon limit. Thus for a long-term investor, it is important to choose a tamed strategy and set the target return rate below the threshold.

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Appendix A: Derivations and Proofs

Derivation of asymptotic behavior of $Brp(T)$ of the MMV strategy given by Eq. (22):

Proof: Equation (21) shows that $Brp(T)$ is only a function of the ratio $\eta = \omega_1/\omega_2$:

$$Brp(T) = 1 - \Phi\left(\frac{1}{\sqrt{T}}\left(\ln\eta + \frac{1}{2}T\right)\right). \tag{A.1}$$

Here η is the root of the following equation, derived from Eq. (19).

$$\frac{\eta\,\Phi\left(\frac{\ln\eta+\frac{1}{2}T}{\sqrt{T}}\right) - \Phi\left(\frac{\ln\eta-\frac{1}{2}T}{\sqrt{T}}\right)}{\eta\,\Phi\left(\frac{\ln\eta-\frac{1}{2}T}{\sqrt{T}}\right) - e^{T}\,\Phi\left(\frac{\ln\eta-\frac{3}{2}T}{\sqrt{T}}\right)} = e^{R_1 T}. \tag{A.2}$$

In order to determine $\lim\limits_{T\to\infty} Brp(T)$, we need to investigate the asymptotic behavior of $\ln\eta(T)$ in the limit $T\to\infty$. We verified that when $T\to\infty$, Eq. (A.2) can not admit a solution of finite value. Therefore, we only need to consider the choices $\lim\limits_{T\to\infty}\ln\eta(T) = \pm\infty$.

We consider the case $\lim\limits_{T\to\infty}\ln\eta(T) = +\infty$ first. To analyze the solution of (A.2) for large T, we need to examine the limit of $\dfrac{\ln\eta(T)-\frac{1}{2}T}{\sqrt{T}}$ for large T. $\Phi(x)$ has the following asymptotic expressions:

$$\Phi(x) = 1 - \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,\frac{1}{x} + O\left(e^{-\frac{x^2}{2}}\,\frac{1}{x^2}\right) \quad \text{as } x\to+\infty.$$

One can show that choices that the limit $\lim\limits_{T\to\infty}\dfrac{\ln\eta(T)-\frac{1}{2}T}{\sqrt{T}} = +\infty$ being finite or positive infinity do not satisfy Eq. (A.2), and the only remaining choice, $\lim\limits_{T\to\infty}\dfrac{\ln\eta(T)-\frac{1}{2}T}{\sqrt{T}} = -\infty$, gives

$$\ln\eta(T) = \left(\frac{1}{2} - \sqrt{2R_1}\right)T + \frac{\ln T}{\sqrt{2R_1}} + o(\ln T), \quad \text{as } T\to\infty. \tag{A.3}$$

Since $\lim\limits_{T\to\infty}\ln\eta(T) = +\infty$, Eq. (A.3) shows that $R_1 \leqslant \frac{1}{8}$.

Now we consider the case $\lim\limits_{T\to\infty}\ln\eta(T) = -\infty$ and examine the limit of $\dfrac{\ln\eta(T)+\frac{1}{2}T}{\sqrt{T}}$ for large T. One can show $\lim\limits_{T\to\infty}\dfrac{\ln\eta(T)+\frac{1}{2}T}{\sqrt{T}}$ being finite is not a solution of Eq. (A.2).

If $\lim\limits_{T\to\infty}\dfrac{\ln\eta(T)+\frac{1}{2}T}{\sqrt{T}} = +\infty$, we obtain

$$\ln\eta(T) = \left(\frac{1}{2} - \sqrt{2R_1}\right)T + \frac{\ln T}{\sqrt{2R_1}} + o(\ln T). \tag{A.4}$$

From the condition $\lim\limits_{T\to\infty}\ln\eta(T) = -\infty$, Eq. (A.4) gives $R_1 > \frac{1}{8}$, and from the condition $\lim\limits_{T\to\infty}\dfrac{\ln\eta(T)+\frac{1}{2}T}{\sqrt{T}} = +\infty$, Eq. (A.4) gives $R \leqslant \frac{1}{2}$. Therefore, in the limit $T\to\infty$,

$$\ln\eta(T) = \left(\frac{1}{2} - \sqrt{2R_1}\right)T + \frac{\ln T}{\sqrt{2R_1}} + o(\ln T), \text{ in the interval } \frac{1}{8} < R \leqslant \frac{1}{2}. \tag{A.5}$$

If $\lim_{T\to\infty} \dfrac{\ln\eta(T)+\frac{1}{2}T}{\sqrt{T}} = -\infty$, we obtain

$$\ln\eta(T) = -R_1 T + o(T), \quad \text{when} \quad R_1 > \frac{1}{2}. \tag{A.6}$$

The relation $R_1 > \frac{1}{2}$ comes from the condition that $\lim_{T\to\infty} \dfrac{\ln\eta(T)+\frac{1}{2}T}{\sqrt{T}} = -\infty$. Equations (A.3), (A.5) and (A.6) can be combined as:

$$T\to\infty : \ln\eta(T) = \begin{cases} \left(\frac{1}{2}-\sqrt{2R_1}\right)T + \frac{\ln T}{\sqrt{2R_1}} + o(\ln T), & R_1 \leqslant \frac{1}{2} \\ -R_1 T + o(T), & R_1 > \frac{1}{2}. \end{cases} \tag{A.7}$$

After substituting Eq. (A.7) into Eq. (A.1), this completes the proof of Eq. (22).

Derivation for $Brp(T)$ of the PUM strategy given by Eq. (35):

Proof: By introducing the symbols $a = R_3 - \frac{R_3^2}{2}$ and $b = R_3$, Eq. (34) can be rewritten as $X(t) = xe^{\left(R_3-\frac{R_3^2}{2}\right)t + R_3 W(t)} == xe^{b(W(t)+\frac{a}{b}t)}$. By the Girsanov theorem, $\widetilde{W}(t) = W(t) + \frac{a}{b}t$ is a Brownian motion under the probability $\widetilde{P}$, where

$$\frac{d\widetilde{P}}{dP} = Z(t) = e^{-\frac{a}{b}W(t)-\frac{1}{2}\frac{a^2}{b^2}t} = e^{-\frac{a}{b}\widetilde{W}(t)+\frac{1}{2}\frac{a^2}{b^2}t}.$$

Using the joint distribution of $(\widetilde{W}(t), \inf_{0\leqslant s\leqslant t}\widetilde{W}(s))$:

$$\widetilde{P}(\widetilde{W}(t)\in dx, \inf_{0\leqslant s\leqslant t}\widetilde{W}(s)\in dy) = \frac{2(x-2y)}{\sqrt{2\pi t^3}}\, e^{-\frac{(2y-x)^2}{2t}}\, dxdy \qquad (y\leqslant x, y\leqslant 0),$$

we can calculate $Brp(T)$ as follows:

$$\begin{aligned} P(\inf_{0\leqslant t\leqslant T} X(t) \leqslant \delta x) &= E\left(1_{\left\{\inf_{0\leqslant t\leqslant T} X(t)\leqslant \delta x\right\}}\right), \\ &= \widetilde{E}\left(1_{\left\{\inf_{0\leqslant t\leqslant T} \widetilde{W}(t)\leqslant \frac{\ln\delta}{b}\right\}} \frac{1}{Z(T)}\right), \\ &= \Phi\left(\frac{\ln\delta - aT}{b\sqrt{T}}\right) + \Phi\left(\frac{\ln\delta + aT}{b\sqrt{T}}\right) e^{\frac{2a\ln\delta}{b^2}}. \end{aligned}$$

Derivation of λ given by Eq. (43):

Proof: In order to get the expression for λ, we need to introduce two untamed strategies. The first one is the Mean-Variance Strategy, which was first proposed and solved in the single-period setting by Markowitz (1952, 1959) in his pioneering work. Then, the model was extended to multi-period by Hakansson (1971) and Grauer and Hakansson (1993), and to continuous-time setting by Zhou and Li (2000). In the work of Zhou and Li (2000), they use the embedding technique and linear-quadratic (LQ) optimal control theory to solve the mean-variance problem in the continuous-time setting. The formulation of this problem is

$$\text{Goal: } \min Var(X(T)) \tag{A.8}$$

$$s.t.\;\; E(X(T)) = xe^{RT},$$

which has the solution

$$\pi(t) = \frac{e^{(R+1)T}-1}{e^T-1}\,x - X(t), \tag{A.9}$$

$$X(t) = x\,\frac{e^{(R+1)T}-1}{e^T-1} + x\,e^{-\frac{3}{2}t-W(t)} - x\,\frac{e^{(R+1)T}-1}{e^T-1}e^{-\frac{3}{2}t-W(t)}. \tag{A.10}$$

The second one is Final Wealth Targeting Strategy. This strategy was first studied by Duffie and Richardson (1991), which is designed for investors who care about the terminal wealth and attempt to minimize its deviation from the target terminal value. The formulation of this problem is

$$\text{Goal: } \min_{\pi}\; E[(X(T) - xe^{RT})^2] \tag{A.11}$$

which has the solution

$$\pi(t) = x\,e^{RT} - X(t), \tag{A.12}$$

$$X(t) = x\,e^{RT} + x\,e^{-\frac{3}{2}t-W(t)} - x\,e^{RT}e^{-\frac{3}{2}t-W(t)}.$$

Denote the target return rate for "Mean-Variance" and "Final Wealth Targeting" by R_{MV} and R_{FWT}, respectively. By equating their optimal policies $\pi(\cdot)$ given by Eqs. (A.9) and (A.12), we obtain the following relation

$$R_{FWT} = \frac{1}{T}\ln\frac{e^{(R_{MV}+1)T}-1}{e^T-1} \tag{A.13}$$

which shows the equivalence between this two strategies.

Since adding the constraint $\{X(t) \geqslant 0 \text{ a.s. } 0 \leqslant t \leqslant T\}$ to both Mean-Variance strategy (A.8) and Final Wealth Targeting strategy (A.11) does not change the equivalence (A.13) between them, and noting the similarity between relations (41) and (A.11), we can obtain the relation (43) as follows:

$$\lambda = xe^{R_{FWT}} = x\,\frac{e^{(R_1+1)T}-1}{e^T-1}$$

Proof of Theorem 1:

Proof: First we show that the MMV strategy and the PUM strategy are the same in the limit $T \to 0$. For a given finite R_1, in the limit $T \to 0$, we can derive the asymptotic behavior of $\eta(T)$ from Eq. (A.2):

$$\eta(T) = 1 + \frac{1}{R_1} + o(1),$$

and the following expressions for $\omega_1(T)$ and $\omega_2(T)$ from Eq. (19):

$$\omega_1(T) = x\ (1 + R_1) + o(1), \quad \omega_2(T) = x\, R_1 + o(1). \tag{A.14}$$

Notice that $0 \leqslant t \leqslant T$, and letting $T \to 0$ leads to $t \to 0$. After substituting Eq. (A.14) into Eqs. (16) and (17) and performing algebraic manipulation, we obtain:

$$\lim_{T\to 0} y(t) \,=\, x\, R_1, \tag{A.15}$$
$$\lim_{T\to 0} d_+(t, y(t)) \,=\, -\infty \quad \text{a.s.} \tag{A.16}$$

From Eqs. (14) and (15), we can rewrite $\pi_1(t)$ as follows:

$$\pi_1(t) = \Phi(-d_+(t, y(t)))\, y(t), \quad 0 \leqslant t \leqslant T. \tag{A.17}$$

After substituting Eqs. (A.15) and (A.16) into Eq. (A.17), we obtain

$$\lim_{T\to 0} \pi_1(t) = x\, R_1, \qquad \text{a.s. } 0 \leqslant t \leqslant T. \tag{A.18}$$

For the PUM strategy, starting from Eqs. (31), (32) and (34), and following some algebraic manipulation, we obtain $\pi_3(t) = x\, R_3 \exp\big((R_3 - \frac{1}{2}R_3)t + R_3\, W(t)\big)$. Since $t \to 0$ as $T \to 0$, we get

$$\lim_{T\to 0} \pi_3(t) = x\, R_3. \tag{A.19}$$

Setting $R_1 = R_3$ and comparing Eq. (A.18) with Eq. (A.19) gives the result:

$$\lim_{T\to 0} \pi_1(t) = \lim_{T\to 0} \pi_3(t).$$

Finally, we show that the MMV strategy and the SPM strategy have the same asymptotic behavior as $T \to \infty$. For a given finite R, from Eq. (22), we know that for the MMV strategy:

$$\begin{aligned} &\lim_{T\to\infty} X(T) > 0 \ \ a.s.. \text{ as } R_1 < \frac{1}{2} = R_1^*, \\ &\lim_{T\to\infty} X(T) = 0 \ \ a.s.. \text{ as } R_1 > \frac{1}{2} = R_1^*. \end{aligned} \tag{A.20}$$

Therefore, as $T \to \infty$ and $R_1 < \frac{1}{2}$, $X(T)$ is positive and can be rewritten as $X(T) = \omega_1 - \omega_2\, e^{-\frac{1}{2}T - W(T)} > 0$, and from Eq. (A.7), $\ln \eta(T)$ has the following asymptotic expression:

$$\ln \eta(T) \,=\, \left(\frac{1}{2} - \sqrt{2R_1}\right) T + o(T). \tag{A.21}$$

We define $\Phi_1 = \Phi\left(\frac{\ln \eta(T) + \frac{1}{2}T}{\sqrt{T}}\right)$ and $\Phi_2 = \Phi\left(\frac{\ln \eta(T) - \frac{1}{2}T}{\sqrt{T}}\right)$. Then, the second equation of Eq. (19) can be written as $x\, e^{R_1 T} = \omega_1\, \Phi_1 - \omega_2\, \Phi_2$, where $\lim\limits_{T\to\infty} \Phi_1 = 1$ and $\lim\limits_{T\to\infty} \Phi_2 = 0$ according to Eq. (A.21). Then, we can derive

$$\lim_{T\to\infty} \frac{X(T)}{x\, e^{R_1 T}} = \lim_{T\to\infty} \frac{\eta(T) - e^{-\frac{1}{2}T - W(T)}}{\eta(T)\, \Phi_1 - \Phi_2} = 1 \quad \text{a.s.} \tag{A.22}$$

Since $X(T)$ of the SPM strategy given by Eq. (27) satisfies the binomial distribution, from Eq. (30), we can derive:

$$\lim_{T\to\infty} \frac{X(T)}{x\, e^{R_2 T}} = 1 \;\; a.s.. \text{ as } R_2 < \frac{1}{2} = R_2^*, \tag{A.23}$$

$$\lim_{T\to\infty} X(T) = 0 \;\; a.s.. \text{ as } R_2 > \frac{1}{2} = R_2^*. \tag{A.24}$$

Setting $R_1 = R_2$ and comparing Eq. (A.22) with Eq. (A.23), and Eq. (A.20) with Eq. (A.24), we have proved that the MMV strategy and the SPM strategy have the same terminal distribution when $T \to \infty$.

Notice that $\lim\limits_{T\to\infty} X(T) \to \infty$ as $R < R^*$. Therefore we denote the wealth $\hat{X}(t) = e^{-Rt}X(t)$ when $R < R^*$, which is uniformly bounded and satisfies the following backward stochastic differential equation:

$$d\hat{X}(t) = (-R\hat{X}(t) + \pi(t))dt + \pi(t)dW(t).$$

Then after applying Proposition 2.2 of El Karoui et al. (1997), we obtain the following unique representations for $X(t)$ and $\hat{X}(t)$

$$\begin{aligned} X(t) &= \rho_1(t)^{-1}E(\rho_1(T)X(T)|\mathcal{F}_t), \;\; \forall t \in [0,T], \; a.s. \text{ as } R < R^*, \\ \hat{X}(t) &= \rho_2(t)^{-1}E(\rho_2(T)\hat{X}(T)|\mathcal{F}_t), \;\; \forall t \in [0,T], \; a.s. \text{ as } R > R^*, \end{aligned}$$

where $\rho_1(t) = e^{-\frac{1}{2}t - W(t)}$ and $\rho_2(t) = e^{(R-\frac{1}{2})t - W(t)}$. Consequently, the equivalence between the MMV strategy and the SPM strategy as $T \to \infty$ is established, namely

$$\lim_{T\to\infty} \pi_1(t) = \lim_{T\to\infty} \pi_2(t).$$

Proof of Theorem 2:

Proof: We will derive the asymptotic expressions of expected utility functions of the MMV and SPM strategies and show that these asymptotic expressions are the same in the limit $T \to \infty$.

For the MMV strategy, from Eq. (20), we can obtain the distribution function and density function of final wealth $X(T)/x$:

$$F_1(z) = P(X(T) \leqslant zx) = \begin{cases} 0, & z < 0 \\ \Phi\left(\frac{1}{\sqrt{T}}\left(\ln\left(\frac{\omega_2}{\omega_1 - z}\right) - \frac{1}{2}T\right)\right), & 0 \leqslant z < \omega_1 \\ 1, & z \geqslant \omega_1, \end{cases}$$

$$\begin{aligned} f_1(z) = \frac{dF_1(z)}{dz} &= \Phi\left(\frac{1}{\sqrt{T}}\left(\ln\left(\frac{\omega_2}{\omega_1}\right) - \frac{1}{2}T\right)\right)\delta(z) + \\ &\phi\left(\frac{1}{\sqrt{T}}\left(\ln\left(\frac{\omega_2}{\omega_1 - z}\right) - \frac{1}{2}T\right)\right)\frac{1}{\omega_1 - z}\frac{1}{\sqrt{T}}\left(H(z) - H(z-\omega_1)\right), \end{aligned}$$

where $\delta(z)$ is the delta function at $z=0$, and $H(z)$ is the Heaviside function defined as:

$$H(z) = \begin{cases} 0, & z<0 \\ 1 & z \geqslant 0. \end{cases}$$

Using the utility function defined in Eq. (42), we have

$$\begin{aligned} E(U_1(X_1(T))) &= \int_{-\infty}^{+\infty} U_1(z) f_1(z) dz, \\ &= 1 - \frac{1}{\lambda^2}\left(\int_{-\infty}^{0} (z-\lambda)^2 f_1(z) dz + \int_0^{\lambda} (z-\lambda)^2 f_1(z) dz\right), \\ &= 1 - F_1(0) - \frac{1}{\lambda^2\sqrt{T}} \int_0^{\lambda} (z-\lambda)^2 \phi\left(\frac{1}{\sqrt{T}}\left(\ln\left(\frac{\omega_2}{\omega_1 - z}\right) - \frac{T}{2}\right)\right) \frac{1}{\omega_1 - z} dz. \quad \text{(A.25)} \end{aligned}$$

By setting $y = \frac{1}{\sqrt{T}} \ln\left(1 - \frac{z}{\omega_1}\right)$, the last term of the above expression can be rewritten as

$$\begin{aligned} &\frac{1}{\lambda^2}\frac{1}{\sqrt{T}} \int_0^{\lambda} (z-\lambda)^2 \phi\left(\frac{1}{\sqrt{T}}\left(\ln\left(\frac{\omega_2}{\omega_1 - z}\right) - \frac{1}{2}T\right)\right) \frac{1}{\omega_1 - z} dz \\ &= \frac{1}{\lambda^2} \int_{\frac{1}{\sqrt{T}}\ln\left(1-\frac{\lambda}{\omega_1}\right)}^{0} \left(\omega_1 e^{\sqrt{T}y} + \lambda - \omega_1\right)^2 \phi\left(y + \frac{1}{\sqrt{T}}\ln\eta(T) + \frac{1}{2}\sqrt{T}\right) dy. \qquad \text{(A.26)} \end{aligned}$$

In the following we will show that Eq. (A.26) always equals to zero in the limit $T \to \infty$.

First, we consider the case $R_1 > \frac{1}{2}$. We show that the integrand Eq. (A.26) is always finite. This is due to the fact $0 \leqslant y \leqslant \frac{1}{\sqrt{T}} \ln\left(1 - \frac{\lambda}{\omega_1}\right)$ which leads to $0 \leqslant \frac{1}{\lambda^2}\left(\omega_1 e^{\sqrt{T}y} + \lambda - \omega_1\right)^2 \leqslant 1$, and the fact $0 \leqslant \phi\left(y + \frac{1}{\sqrt{T}}\ln\eta(T) + \frac{1}{2}\sqrt{T}\right) \leqslant \frac{1}{\sqrt{2\pi}}$. Now we show that the lower limit of integration in (A.26) tends to zero as $T \to \infty$.

From Eq. (A.7) we have the following asymptotic behavior of $\ln\eta(T)$

$$\ln\eta(T) = -R_1 T + o(T), \text{ as } T \to \infty \qquad \text{(A.27)}$$

and it follows

$$\eta(T) = C_1(T)\, e^{-R_1 T}, \text{ as } T \to \infty, \qquad \text{(A.28)}$$

where $C_1(T) = e^{o(T)}$.

Notice that $\omega_1 = \eta\,\omega_2$. It is easy to derive the expressions for ω_1 and ω_2 in terms of η from Eq. (19):

$$\omega_1 = x\, \frac{\eta\, e^{R_1 T}}{\eta\,\Phi_1 - \Phi_2} \quad \text{and} \quad \omega_2 = x\, \frac{e^{R_1 T}}{\eta\,\Phi_1 - \Phi_2},$$

where $\Phi_1 = \Phi\left(\frac{\ln\eta(T) + \frac{1}{2}T}{\sqrt{T}}\right)$ and $\Phi_2 = \Phi\left(\frac{\ln\eta(T) - \frac{1}{2}T}{\sqrt{T}}\right)$. Then, using Eqs. (A.27) and (A.28), we have

$$\lim_{T\to\infty} \frac{\lambda}{\omega_1} = \lim_{T\to\infty} \frac{\eta\,\Phi_1 - \Phi_2}{\eta} \frac{e^{(R_1+1)T} - 1}{e^{R_1 T}(e^T - 1)} = \lim_{T\to\infty} \Phi_1 - \Phi_2\,\eta^{-1} = 0. \qquad \text{(A.29)}$$

This leads to $\lim_{T\to\infty} \frac{1}{\sqrt{T}} \ln\left(1 - \frac{\lambda}{\omega_1}\right) = 0$. Therefore Eq. (A.26) equals to zero in the limit $T \to \infty$ since $R > \frac{1}{2}$.

To analyze the case $R \leqslant \frac{1}{2}$, we derive the closed expression for Eq. (A.26) and the result can be expressed as

$$(\text{A.26}) = A_1 + A_2 + A_3,$$

where

$$\begin{aligned}
A_1 &= \left(\frac{\omega_2}{\lambda}\right)^2 e^T \left[\Phi\left(\frac{\ln(\omega_1/\omega_2) - \frac{3}{2}T}{\sqrt{T}}\right) - \Phi\left(\frac{\ln((\omega_1-\lambda)/\omega_2) - \frac{3}{2}T}{\sqrt{T}}\right)\right],\\
A_2 &= \frac{2\omega_2}{\lambda}\left(1-\frac{\omega_1}{\lambda}\right)\left[\Phi\left(\frac{\ln(\omega_1/\omega_2) - \frac{1}{2}T}{\sqrt{T}}\right) - \Phi\left(\frac{\ln((\omega_1-\lambda)/\omega_2) - \frac{1}{2}T}{\sqrt{T}}\right)\right],\\
A_3 &= \left(1-\frac{\omega_1}{\lambda}\right)^2\left[\Phi\left(\frac{\ln(\omega_1/\omega_2) + \frac{1}{2}T}{\sqrt{T}}\right) - \Phi\left(\frac{\ln((\omega_1-\lambda)/\omega_2) + \frac{1}{2}T}{\sqrt{T}}\right)\right].
\end{aligned}$$

Since $R_1 \leqslant \frac{1}{2}$, from Eq. (A.7), we have the following asymptotic behavior of $\ln\eta(T)$:

$$\ln\eta(T) = \left(\frac{1}{2} - \sqrt{2R_1}\right)T + \frac{\ln T}{\sqrt{2R_1}} + o(\ln T), \text{ as } T \to \infty. \tag{A.30}$$

Consequently,

$$\eta(T) = C_2(T)\, e^{\left(\frac{1}{2}-\sqrt{2R_1}\right)T + \frac{\ln T}{\sqrt{2R_1}}}, \text{ as } T \to \infty, \tag{A.31}$$

where $C_2(T) = e^{o(\ln T)}$. Then, using Eqs. (A.30) and (A.31), and following a procedure similar to what we did in Eq. (A.29), we have

$$\lim_{T\to\infty}\frac{\lambda}{\omega_1} = \begin{cases} 1, & R_1 < \frac{1}{2} \\ \frac{1}{2}, & R_1 = \frac{1}{2}. \end{cases} \tag{A.32}$$

Notice that all second terms in A_1, A_2 and A_3 contain the factor

$$\ln((\omega_1-\lambda)/\omega_2) = \ln(\omega_1/\omega_2) + \ln(1 - \frac{\lambda}{\omega_1}), \tag{A.33}$$

and Eqs. (A.32) and (A.33) lead to

$$\lim_{T\to\infty}\ln\left(1-\frac{\lambda}{\omega_1}\right) = \begin{cases} -\infty, & R_1 < \frac{1}{2} \\ -\ln 2, & R_1 = \frac{1}{2}. \end{cases}$$

This means that the second terms in A_1, A_2 and A_3 are of same order or of higher order relative to the corresponding first term.

For calculating A_1 and A_2, we analyze their first terms in the limit $T \to \infty$. By using Eqs. (A.30), (A.31) and (A.32), it is not difficult to show

$$\begin{aligned}
&\lim_{T\to\infty}\left(\frac{\omega_2}{\lambda}\right)^2 e^T\,\Phi\left(\frac{\ln(\omega_1/\omega_2) - \frac{3}{2}T}{\sqrt{T}}\right) = 0,\\
&\lim_{T\to\infty}\frac{2\omega_2}{\lambda}\left(1-\frac{\omega_1}{\lambda}\right)\Phi\left(\frac{\ln(\omega_1/\omega_2) - \frac{1}{2}T}{\sqrt{T}}\right) = 0.
\end{aligned}$$

Therefore $\lim\limits_{T\to\infty} A_1 = \lim\limits_{T\to\infty} A_2 = 0$.

For determining the asymptotic value of A_3, we notice that

$$\lim_{T\to\infty}\left(1-\frac{\omega_1}{\lambda}\right)^2 = 0 \text{ as } R_1 < \frac{1}{2}$$
$$\lim_{T\to\infty}\Phi\left(\frac{\ln(\omega_1/\omega_2)+\frac{1}{2}T}{\sqrt{T}}\right)-\Phi\left(\frac{\ln((\omega_1-\lambda)/\omega_2)+\frac{1}{2}T}{\sqrt{T}}\right)=0 \text{ as } R_1=\frac{1}{2}.$$

Thus, we have $A_3 = 0$ in the limit $T\to\infty$ for $R_1 \leqslant \frac{1}{2}$. Therefore, Eq. (A.26) always equals to zero in the limit $T\to\infty$ and Eq. (A.25) becomes $1-F_1(0)$. Then, the corresponding result for the MMV strategy in Eq. (48) can be easily obtained.

For the SPM strategy, $X(T)$ satisfies the binomial distribution shown in Eq. (27) which gives us $F_2(z) = (1-P)\,H(z) + P\,H(z-e^{R_2T})$, where $P = \Phi\left(\sqrt{T}+\Phi^{-1}\left(e^{-R_2T}\right)\right)$. Using the utility function defined in Eq. (44), we have

$$E(U_2(X_2(T))) = \int_{e^{R_2T}}^{+\infty} dF_2(z) = \Phi\left(\sqrt{T}+\Phi^{-1}\left(e^{-R_2T}\right)\right).$$

From the asymptotic expansions of $\Phi^{-1}(\cdot)$ given by Eq. (29), the corresponding result for the SPM strategy in Eq. (48) can be easily obtained.

Appendix B: Resolving a Paradox

In this appendix, we show how the condition $E(X(T)) = xe^{RT}$ is satisfied when the investment leads to sure bankruptcy. We use the PUM strategy as an example to explain this paradox.

From the wealth process Eq. (34), we can get the cumulative distribution function of $X(T)/x$ as follows:

$$\begin{aligned} F_{X_T}(z) &= P(X(T)\leqslant zx) \\ &= \Phi\left(\frac{1}{R_3\sqrt{T}}\ln z-\left(1-\frac{R_3}{2}\right)\sqrt{T}\right), \quad 0<z<+\infty. \end{aligned} \tag{B.1}$$

We now examine the conditional expected wealth for the paths with wealth larger than a given level Cx and the probability that such paths occur. These quantities can easily be derived from Eq. (B.1):

$$P(X(T)\geqslant Cx) = 1-\Phi\left(\frac{1}{R_3\sqrt{T}}\ln C-(1-\frac{R_3}{2})\sqrt{T}\right), \tag{B.2}$$

$$E(X(T)\,1_{\{X(T)\geqslant Cx\}}) = e^{R_3T}\left(1-\Phi\left(\frac{1}{R_3\sqrt{T}}\ln C-(1+\frac{R_3}{2})\sqrt{T}\right)\right). \tag{B.3}$$

Equations (B.2) and (B.3) lead to the following conditional expectation:

$$\begin{aligned} E(X(T)|X(T)\geqslant Cx) &= \frac{E(X(T)\,1_{\{X(T)\geqslant Cx\}})}{P(X(T)\geqslant Cx)} \\ &= e^{R_3T}\,\frac{1-\Phi\left(\frac{1}{R_3\sqrt{T}}\ln C-(1+\frac{R_3}{2})\sqrt{T}\right)}{1-\Phi\left(\frac{1}{R_3\sqrt{T}}\ln C-(1-\frac{R_3}{2})\sqrt{T}\right)}. \end{aligned} \tag{B.4}$$

As $T \to \infty$ and $R_3 > R_3^* = 2$, Eq. (B.2) shows that the probability of the final wealth being above the level Cx decreases to 0 for any positive C, while Eq. (B.4) shows that the conditional expectation of the wealth being above the level Cx increases much faster than $e^{R_3 T}$ due to the fact:

$$\lim_{T\to\infty} \frac{1-\Phi\left(\frac{1}{R_3\sqrt{T}}\ln C-(1+\frac{R_3}{2})\sqrt{T}\right)}{1-\Phi\left(\frac{1}{R_3\sqrt{T}}\ln C-(1-\frac{R_3}{2})\sqrt{T}\right)}=\infty.$$

This can be easily understood if we choose $C \triangleq C(T) = e^{R_3 T}$, the expectation of the final wealth. Then Eqs. (B.2) and (B.4) become:

$$P(X(T) \geqslant xe^{R_3 T}) = 1-\Phi\left(\frac{R_3}{2}\sqrt{T}\right), \tag{B.5}$$

$$E(X(T)|X(T) \geqslant xe^{R_3 T}) = e^{R_3 T}\frac{\Phi\left(\frac{R_3}{2}\sqrt{T}\right)}{1-\Phi\left(\frac{R_3}{2}\sqrt{T}\right)}. \tag{B.6}$$

Equations (B.5) and (B.6) give a simple explanation to the paradox: as T increases, although the fraction of paths ending above the target becomes very small, the final wealth of those paths that do end above the target becomes much larger than the expected target wealth such that the condition $E(X(T)) = xe^{RT}$ still holds. Of course, the probability for an investor to obtain this wealth is really small.

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