Derivatives Pricing and Trading in Incomplete Markets

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Outline

Describe THE Idea

The Risky Bond Example
Incomplete Market Models

Model:

▶ Abstraction of reality
▶ Simulated option game
▶ No absolute correctness in finance

What are the logical consequences after establishing a belief?
Incomplete Market Models

Model:
- Abstraction of reality
- Simulated option game
- No absolute correctness in finance

What are the logical consequences after establishing a belief?

Incomplete Markets:
Cannot eliminate risks associated with a derivative position.

Causes for Incompleteness:
Transaction costs, Stochastic vloatility, Jumps, Trading contraints, etc.

Reality is much better represented by incomplete markets.
Preference Question

Why is it necessary?

- The final wealth is a random variable.
- Different strategies (e.g. hedging schemes) produce different probability density functions of the final wealth.
- **Must** find a way to rank different strategies.

Example:
Strategy A: a Gaussian with mean 1.0, standard deviation 1.0;
Strategy B: a Gaussian with mean 0.5, standard deviation 0.4.
Which one do you choose?
Utility Function

Standard approach is the expected utility theory

\[ E[U] = \int U(w) \rho(w) \, dw \]

Change \( \int \) to \( \sum \) if \( w \) is discrete.

\( U(w) \) is increasing and concave.
Affine transformation freedom of utility functions.
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Use the negative exponential utility function

\[ U(w) = -\frac{1}{\gamma} \exp(-\gamma w) \]

Large risk aversion parameter \( \gamma \) means more risk averse.
\( \gamma \) and position size appear together as a product.

Reason: Memoryless, Solvable
Fair Value

Fair value is the model output price of a derivative contract.
Fair Value

Fair value is the model output price of a derivative contract.

How to use your fair value $f$:

if $p < f$, you buy;
if $p = f$, you hold;
if $p > f$, you sell;

where $p$ is the market price of the derivative.
Review

The “Aha!” moment is coming up soon.
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Four ingredients:

- Logic
The “Aha!” moment is coming up soon.

Four ingredients:
- Logic
- Incomplete market model
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Four ingredients:
- Logic
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- Utility function
The “Aha!” moment is coming up soon.

Four ingredients:

- Logic
- Incomplete market model
- Utility function
- Notion of fair value
Aha!

In a local equilibrium when $p = f$. 

The equilibrium state is optimal!
In a local equilibrium when $p = f$.

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The equilibrium state is optimal!

Aha! The link: derivative pricing and portfolio optimization
Aha!

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The equilibrium state is optimal!

Aha! The link: derivative pricing and portfolio optimization

What are the necessary conditions for optimality?

$\implies$ Equations for computing the fair value
New Pricing Principle

Local Equilibrium Principle > Arbitrage Principle
New Pricing Principle

<table>
<thead>
<tr>
<th>Local Equilibrium Principle &gt; Arbitrage Principle</th>
</tr>
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<tbody>
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<td>Complete</td>
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</tbody>
</table>
New Pricing Principle

Local Equilibrium Principle $>$ Arbitrage Principle

Complete
- Local equilibrium pricing
- delta hedging & BS eq.

Incomplete
- unique and correct

Arbitrage pricing
- delta hedging $\Rightarrow$ BS eq.
- a very wide range

Explicit link: Real measure $\longrightarrow$ Pricing measure

Warning: No more freedom to yank a “risk neutral” measure out of thin air, i.e. cannot model “risk neutral” measure directly.
Outline

Describe THE Idea

The Risky Bond Example
Model

- unit face value zero-coupon bond maturing at time $T$
- probability of default is $d$
- zero interest rate and other idealized assumptions
- current market price of the illiquid risky bond is $p$

This is an incomplete market model.

The risky bond is considered as a derivative here.

This simplest financial model goes a long way to explain all the relevant concepts.

**Goal:** systematic trading decisions based on the model
Portfolio Optimization

The expected utility of the final wealth is

\[ E[U] = (1 - d) \ U(w_0 + (1 - p)\hat{n}) + d \ U(w_0 - p\hat{n}) \]

Set the first order derivative w.r.t. \( \hat{n} \) to zero

\[(1 - d)(1 - p) \ U'(w_0 + (1 - p)\hat{n}) = dp \ U'(w_0 - p\hat{n})\]
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\]

The optimal position is (no \( w_0 \))

\[
\gamma \hat{n} = \ln \left( \frac{(1 - d)(1 - p)}{dp} \right)
\]
Fair Value

Let $n$ be your current position, your fair value of the risky bond is

$$f = \frac{1 - d}{(1 - d) + d \exp(\gamma n)}$$

Inversion:
What market price makes the current position optimal?
Fair Value

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Inversion:
What market price makes the current position optimal?

Proof:
- if $p < f$, then $\hat{n} > n$, $\Rightarrow$ you buy;
- if $p = f$, then $\hat{n} = n$, $\Rightarrow$ you hold;
- if $p > f$, then $\hat{n} < n$, $\Rightarrow$ you sell.
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$f$ depends on the model parameter $d$—no surprise.
$f$ also depends on your risk preference $\gamma$ and current position $n$!
Fair Value

Let \( n \) be your current position, your fair value of the risky bond is

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What market price makes the current position optimal?

Proof:

\begin{itemize}
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\end{itemize}

\( f \) depends on the model parameter \( d \)—no surprise. \( f \) also depends on your risk preference \( \gamma \) and current position \( n \)!

The fair value concept is only meaningful when you take the personal rather than the market perspective.
Source of Risk

Incomplete markets $\Rightarrow$ Unhedgable Risks
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Q: What is the source of the risk?
A: Having a position (your position!).

Incompleteness + Risk Aversion $\Rightarrow$ Position Dependency
Source of Risk

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Current Literature:
Missing Position Dependency $=$ Missing Risks
Source of Risk

Incomplete markets ⇒ Unhedgable Risks

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A: Having a position (your position!).

Incompleteness + Risk Aversion ⇒ Position Dependency

Current Literature:
Missing Position Dependency = Missing Risks

The position effect can offer natural explanations to many real world phenomenons.
How to Trade

Position dependency $f(n) \Rightarrow$ Natural trading strategy

Trading Rule: (do not require gut feelings)
Make post-trade fair value equal the market price

$$f(n + m) = p$$

This is the local equilibrium equation.
How to Trade

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f(n + m) = p
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This is the local equilibrium equation.

The solution is

\[
m = \frac{1}{\gamma} \ln \frac{(1 - d)(1 - p)}{dp} - n = \hat{n} - n
\]

The optimal trading size \( m \) is simply the optimal position \( \hat{n} \) (post-trade) minus the current position \( n \) (pre-trade).
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Incomplete Market Model + Risk Aversion = How to Trade
Define a curve $q(m) := f(n + m)$

$$q(m) = \frac{1 - d}{(1 - d) + d \exp[\gamma(n + m)]}$$

$d = 0.05, \gamma n = 0.5$
Define a curve $q(m) := f(n + m)$

$$q(m) = \frac{1 - d}{(1 - d) + d \exp[\gamma(n + m)]}$$

$p < f(n) \Rightarrow m > 0$ (demand)

$p > f(n) \Rightarrow m < 0$ (supply)

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large $|p - f(n)| \Rightarrow$ large $|m|$

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\( p < f(n) \Rightarrow m > 0 \) (demand)

\( p > f(n) \Rightarrow m < 0 \) (supply)

large \( |p - f(n)| \Rightarrow \) large \( |m| \)

downward sloping guarantees equilibrium state

automatic inventory control
Generating Quotes

The personal supply-demand curve is also called quote price curve.
Generating Quotes

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Let $m_b > 0$ (bid) and $m_a < 0$ (ask)

Making a market: Posting four numbers

$\{q(m_b), |m_b|\} - \{q(m_a), |m_a|\}$, \textit{e.g.}, $\{0.875, 0.5\} - \{0.950, 0.5\}$

$\{\text{bid price, bid size}\} - \{\text{ask price, ask size}\}$
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**Making a market:** Posting four numbers

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{bid price, bid size}—{ask price, ask size}

Natural market maker!
Arbitrage Price

Definition for buy and sell arbitrage prices (Why?)

\[ a^b := \lim_{m \to +\infty} f(n + m) \]
\[ a^s := \lim_{m \to -\infty} f(n + m) \]

\( a^b \) and \( a^s \) are position and preference independent.
Arbitrage Price

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\(a^b\) and \(a^s\) are position and preference independent.

Arbitrage prices are not useful in incomplete markets because \((a^b, a^s)\) form a wide range.

For the risky bond, \(a^b = 0\) and \(a^s = 1\).
Certainty Equivalent Profit and Loss (CEPL)

How to measure a trade?

- Realized P&L: a random ex-post quantity
- Gain in expected utility: no natural scale
- **CEPL**: convert expected utility gain into wealth

Trading \( m \) units at \( p \) per bond:

\[
E_1[U] = (1 - d)U(w_0 - pm + n + m) + dU(w_0 - pm)
\]

Taking the lump sum \( \Upsilon \) in lieu of the trade:

\[
E_2[U] = (1 - d)U(w_0 + \Upsilon + n) + dU(w_0 + \Upsilon)
\]

CEPL definition:

\[
\text{Indifferent} \iff E_1[U] = E_2[U]
\]

\[
\Upsilon(m, p) = -\frac{1}{\gamma} \ln d + (1 - d) \exp[-\gamma(m + n)]d + (1 - d) \exp[-\gamma n] - mp
\]
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Taking the lump sum $\Upsilon$ in lieu of the trade:

$$E_2[U] = (1 - d) \ U(w_0 + \Upsilon + n) + d\ U(w_0 + \Upsilon)$$

**CEPL definition**: Indifferent $\Rightarrow E_1[U] = E_2[U]$

$$\Upsilon(m, p) = -\frac{1}{\gamma} \ln \frac{d + (1 - d) \exp[-\gamma(m + n)]}{d + (1 - d) \exp(-\gamma n)} - mp$$
Dimensionless CEPL Surface $\gamma \Upsilon(m, p)$
CEPL against Trading Price

Sideway view of the surface plot
CEPL against Trading Size

Front view of the surface plot
Portfolio Indifference Price

Indifferent between lump sum $h$ and position $n$

$$U(w_0 + h) = (1 - d) \ U(w_0 + n) + d \ U(w_0)$$

Explicit formula

$$h(n) = -\frac{1}{\gamma} \ln \left[ d + (1 - d) \exp(-\gamma n) \right]$$

Note $n = 0 \Rightarrow h = 0$. 
Portfolio Indifference Price

Indifferent between lump sum $h$ and position $n$

$$U(w_0 + h) = (1 - d) \ U(w_0 + n) + d \ U(w_0)$$

Explicit formula

$$h(n) = -\frac{1}{\gamma} \ln \left[ d + (1 - d) \exp(-\gamma n) \right]$$

Note $n = 0 \Rightarrow h = 0$.

The CEPL formula can be rewritten as

$$\Upsilon(m, p) = h(m + n) - h(n) - mp$$

Can be deduced from the notion of indifference.
Tangent Relation

\[ f(n) = h'(n) \]

Easy proof mathematically
Tangent Relation

\[ f(n) = h'(n) \]

Easy proof mathematically

Two proofs based on financial interpretations:

Proof #1: Infinitesimal trade after establishing equilibrium

\[ 0 = h(\epsilon + n) - h(n) - \epsilon p \]
Tangent Relation

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Easy proof mathematically

Two proofs based on financial interpretations:

Proof #1: Infinitesimal trade after establishing equilibrium

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Proof #2: Establishing the optimal position from nothing

\[ \Upsilon = h(n) - np \]
Tangent Relation

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Easy proof mathematically

Two proofs based on financial interpretations:

Proof #1: Infinitesimal trade after establishing equilibrium

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Proof #2: Establishing the optimal position from nothing

\[ \Upsilon = h(n) - np \]

Concavity of \( h(n) \) \( \Rightarrow \) downward slope of \( f(n) \)
Reserve Price

Why needed? Trading size not infinitely divisible.

Another CEPL formula: (easy financial interpretation)

\[ \Upsilon(m, p) = m \left[ r(m) - p \right] \]

Negative CEPL if \( r_b(\mid m \mid) < p < r_s(\mid m \mid) \).

Optimal CEPL formula:

\[ \Upsilon_0(m) = m \left[ r(m) - q(m) \right] \geq 0 \]

\( \Upsilon_0(m) \leftarrow \text{quote price curve} q(m) \rightarrow \Upsilon_0(p - f(n)) \)
Reserve Price

Why needed? Trading size not infinitely divisible.
Zero CEPL if trading $m$ units at $r(m)$ per unit

$$r(m) = \frac{1}{m} [h(n + m) - h(n)]$$

$$= \frac{1}{\gamma m} \ln \frac{d + (1 - d) \exp(-\gamma n)}{d + (1 - d) \exp[-\gamma(n + m)]}$$
Reserve Price

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= \frac{1}{\gamma m} \ln \frac{d + (1 - d) \exp(-\gamma n)}{d + (1 - d) \exp[-\gamma(n + m)]}
\]

Another CEPL formula: (easy financial interpretation)

\[
\gamma(m, p) = m [r(m) - p]
\]

Negative CEPL if \( r^b(|m|) < p < r^s(|m|) \).

Optimal CEPL formula:

\[
\gamma_o(m) = m [r(m) - q(m)] \geq 0
\]

\( \gamma_o(m) \leftarrow \text{quote price curve } q(m) \rightarrow \gamma_o(p - f(n)) \)
Schematic Drawing

- sell arbitrage price $a_s$
- sell quote price $q^s$
- sell reserve price $r^s$
- current fair value $f$
- buy reserve price $r^b$
- buy quote price $q^b$
- buy arbitrage price $a^b$

 Meaning w.r.t. trading size
sell arbitrage price $a^s$
sell quote price $q^s$
sell reserve price $r^s$
current fair value $f$
buy reserve price $r^b$
buy quote price $q^b$
buy arbitrage price $a^b$

Meaning w.r.t. trading size
Intuitive ranking

$r^b(\mu) \approx \frac{1}{2}[q^b(\mu) + q^b(0)]$
sell arbitrage price $a^s$

sell quote price $q^s$

sell reserve price $r^s$

current fair value $f$

buy reserve price $r^b$

buy quote price $q^b$

buy arbitrage price $a^b$

- Meaning w.r.t. trading size
- Intuitive ranking
- $q(m)$ and $r(m)$ asymmetric w.r.t. current fair value
sell arbitrage price $a^s$

sell quote price $q^s$

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- Meaning w.r.t. trading size
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- $q(m)$ and $r(m)$ asymmetric w.r.t. current fair value
- $r(m) \approx \frac{1}{2} [q(m) + q(0)]$
Quote and Reserve Price Curves

Basis for making rational trading decisions!
Mutually Beneficial Trading

Example: Same everything except initial position

\[
\begin{array}{cccc}
\gamma n & \text{c.f.v.} & \gamma m & \text{p.t.f.v} \\
\hline
\text{Trader A} & 0.0 & 0.9500 & 0.25 & 0.9367 & 1.724 \times 10^{-3} \\
\text{Trader B} & 0.5 & 0.9202 & -0.25 & 0.9367 & 1.994 \times 10^{-3} \\
\end{array}
\]

Economical Reason: Risk Transfer!
Mutually Beneficial Trading

Example: Same everything except initial position

<table>
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<th></th>
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Economical Reason: Risk Transfer!

Local Equilibrium: There exists a local equilibrium for any two traders, i.e., one can find a trading size $m_*$ such that

$$f(n + m_*) = \tilde{f}(\tilde{n} - m_*)$$
Mutually Beneficial Trading

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\[
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Global Equilibrium: There exists a global equilibrium state for \( M \) traders.

May not reach there in a reasonable amount of time!
Summary

- derivatives should be priced in the context of portfolio optimization;
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- derivatives pricing is preference and position dependent in incomplete markets, which is only meaningful from the personal perspective;
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- the position dependent pricing offers a natural and systematic way to trade derivatives;
Summary

- derivatives should be priced in the context of portfolio optimization;
- derivatives pricing is preference and position dependent in incomplete markets, which is only meaningful from the personal perspective;
- the position dependent pricing offers a natural and systematic way to trade derivatives;
- derivatives trading in incomplete markets is mutually beneficial.
Further Information: www.atmif.com/qsdt

Quantitative Strategies for Derivatives Trading

Dennis Yang

- Book Excerpt
- Derivatives Pricing and Trading in Incomplete Markets: A Tutorial on Concepts
- A Simple Jump to Default Model