A Note on Risk Neutral Pricing in Incomplete Markets

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Abstract

In the article "Beyond Black-Scholes Option Theory" [5], I questioned the meaning of risk neutral option pricing in incomplete markets, but did not provide an answer. Risk neutral pricing in incomplete markets does have a well-defined, albeit somewhat trivial, meaning. The purpose of this note is to clarify this point, as well as to explain a few related facts.

I Meaning of Risk Neutral

Without abusing the English language, the phrase "risk neutral" can only reasonably be interpreted in two different ways: One is that the the attitude towards the risk is neutral, in the sense of neither risk seeking nor risk averse. Most people and almost all financial institutions have risk averse preference. The other meaning is not having any risk, in the sense that the future wealth is a certainty, which is what "risk neutral" means in the rest of this note. Not taking position in any risky asset is a trivial way of being risk neutral.

These two interpretations of "risk neutral" converge in the limit of infinite capital base. Because an entity with infinite capital has risk neutral preference, and that any finite position size is like having no position at all in the infinite capital limit.

II Complete and Incomplete Markets

Complete and incomplete markets refer to two different types of quantitative finance models. To have risks in a model, you need two ingredients: First, there must be some randomness in the model world. Second, you must take positions in risky assets; without a position, you have no risk, which is quite obvious.

The aforementioned are necessary conditions for having risks, are they sufficient? Let us look at an example, suppose you are short a European call option, long a European put option with the same maturity and strike, and long a share of the underlying asset. No matter how you model the underlying asset, such a position has no risk due to the put-call parity rule. This model independent static cancellation of position risk is obvious. The question is whether there are nontrivial examples in which derivative positions in a stochastic environment carry no risk. The answer is yes, but this requires a very special type of model—the so called complete market model.

In complete markets, the risk of a derivative position can be neutralized through a special trading strategy—delta hedging. In other words, derivatives in complete markets can be replicated through dynamic tradings of the underlying. This fact is by no meaning obvious (worthy of a Nobel prize). However, complete market models are extremely fragile; Fischer Black said himself that the Black-Scholes equation was derived based on ten *unrealistic* assumptions [1], which is another way of saying complete markets do not resemble reality at all.

Incomplete markets, in which positions and risks are inseparable, are much more general, and can be made realistic. The no-arbitrage requirement, which is sufficient for option pricing in complete markets, is too weak in incomplete markets; thus it is unable to uniquely price options. The local-equilibrium principle [5] provides the foundation for a quantitative option theory, which is position dependent. The reason positions play a role in option valuation is that risks matter for risk averse people/institutions, and risks come from positions. Note the local-equilibrium principle is consistent with the no-arbitrage principle, so no matter what the current position is, the pricing result is always arbitrage free.

If option pricing under the local-equilibrium principle is position independent, then the model must belong to complete markets. Said differently, in complete markets, option pricing depends only on the model; but in incomplete markets, it depends on both the model and the current position. I emphasize that the incomplete market option theory derived from the local-equilibrium principle does not rely on the risk neutral concept, in fact it is risk based. The risks from the existing inventory (current position) as well as the new position you plan to take are handled in a natural and systematic way.

By definition, derivative positions cannot be replicated in incomplete markets, thus the phrase "risk neutral" can only be interpreted as having no position at all (hence no risk). There are two types of positions: the underlying (directional bets), and the related derivatives. To understand non-risk-neutral option pricing, the effects of these two types of positions are examined next.

III Role of the Underlying

I first examine the relation of directional bets and option pricing. In complete markets, there must be an underlying asset that can be traded continuously (without transaction costs); but incomplete markets do not have such a requirement. Thus this section is only applicable to those incomplete markets that contain a continuously tradable underlying asset.

Before dealing with derivatives, you should know how to trade the underlying just by itself, which depends crucially on what you think the growth rate (drift) of the underlying asset is. Since the drift term is difficult to estimate accurately in reality, it is your forecast or guess that really matters. You will make some directional bets if you believe the drift differs from the riskless rate.

By the magic of delta hedging, the drift has no influence on option pricing in complete markets, which means the directional trading problem and the option pricing problem are completely separable. Unfortunately such a separation property does not extend to incomplete market models, *i.e.*, whether you make a directional bet influences option pricing. For specific examples, see [4] for the jump to default model, and Chapter 7 of [2] for stochastic volatility models. Therefore in incomplete markets, if two people have different opinions on the drift of the underlying asset, then their respective fair values of an option will differ, assuming everything else being the same. This valuation disparity can be eliminated through a mutually beneficial transaction between them [2, 3].

In complete markets, using risk neutral pricing usually means replacing the drift by the riskless rate in the option pricing equation (Black-Scholes equation). Can this procedure ever be justified in incomplete markets? The answer is yes, under the following two conditions: (i) directional neutral, (ii) no option inventory (to be discussed in the next section). Note the rationals of using the riskless rate is completely different: in complete markets, you use it regardless of what the drift is; whereas in incomplete markets, you use it only when you believe the drift (the actual "physical measure") equals the riskless rate.

Another related problem is what drift to use in the option pricing equation when you have a strong opinion on taking a directional bet, but are constrained to be directional neutral. The answer is to change your personal opinion on the drift, use the riskless rate instead. Otherwise the system will make you take directional bets using options instead of the underlying, which again will be a violation of your directive.

IV Effects from Derivative Positions

I now investigate how derivative positions effect option pricing. As indicated before, it is the position dependent nature that distinguishes option pricing in incomplete markets from that of complete markets. The position dependency causes option pricing in incomplete markets to be nonlinear (with respect to the payoff); at the same time it provides a natural way to trade options via personal supply-demand curves [2, 3].

No inventory is a very special position. In this situation, the derivative position effect disappears, which makes the option pricing equation linear again, as in the case of complete markets. If in addition you are directional neutral, then your current portfolio is empty (no underlying and no derivatives). Option pricing in incomplete markets under this special condition can be regarded as risk neutral, as you have no existing risks. Traditional arbitrage-free derivation of option pricing equation for incomplete markets leaves the so-called market-price-of-risk term unspecified; whereas the local-equilibrium principle based theory can identify this risk related term. Moreover, it can be shown that the market-price-of-risk term is exactly zero in the special case risk neutral pricing (see Section 7.4 of [2]).

How important is the position effect? If the position effect is small, then the risk neutral price can be regarded as a good approximation to position dependent fair values. The simplest financial model on a unit face value risky bond [3] offers a quick answer to this question. From formula (4) in [3], you see that the position effect can make a bond's fair value go from zero to one, while keeping the underlying model (default probability) unchanged. Therefore the position effect can play a dominant role in option pricing. Let me offer another argument on why the position effect is important in practice. Suppose the position effect changes the fare value by five percent from the risk neutral value, which may seem to be small by itself. However if the bid-ask spread of the option is on the order of five percent, then such a shift in the fair value may cause you to go from being a buyer to being a seller, and vice versa.

V Market Calibration

Market calibration is the current mainstream practical method for pricing exotic options, in which models are adjusted such that their outputs for vanilla options match those of market prices. Many authors in the literature simply claim that they are using the "risk neutral" pricing measure when doing market calibrations. I think this is incorrect, as by definition derivative risks cannot be neutralized in incomplete markets. Since market calibration is not logically consistent (see below), it is not possible to demonstrate step by step how the final wealth uncertainty can be eliminated.

I now outline why the market calibration framework is unsatisfactory (at least to me). (i) The model changes after each calibration, which violates what it means to be a model (logical inconsistency). (ii) The distinction between vanilla and exotic options is purely semantic in complete markets, as they are both priced by the Black-Scholes equation. But the market calibration framework breaks the symmetry by drawing a sharp line between the two. (iii) How can a successful theory fail to explain the basics (*i.e.*, fail to price vanilla options)? (iv) It relies on the market simultaneously being both efficient and inefficient. Clearly you would not want to fit to something that you think is wrong. So why are market prices for vanilla options always "right"? The answer is that the market is efficient. Yet at the same time the over-the-counter exotic options market is assumed to be inefficient. The implication is that option pricing theories are useless in a hypothetic world where every derivative contract is traded on an exchange. If you are also bothered by these issues associated with the market calibration framework, then let me point out that none of these concerns arises in the local equilibrium principle based option theory.

In practice, pricing via market calibration is effective [5], but the bottom line is that it is nothing but an ad hoc interpolation scheme. Taking a look at the bigger picture, pricing is only one part in a three-part system, with the other two being trading and risk management. Trading is usually done by traders who use ad hoc subjective judgments; risk management also contains many ad hoc procedures; sometimes risk management and pricing even adopt different models. To summarize, current real-life derivative businesses for many financial institutions are based on a system of three ad hoc components glued together. In contrast, the option theory based on the local-equilibrium principle combines these three separate parts (pricing, trading and risk management) into a single coherent framework. Please kindly re-read the previous sentence, as it highlights the difference between the new approach and the current mainstream practice.

VI Conclusion

Risk neutral pricing in incomplete markets is only meaningful when your current portfolio is empty, *i.e.*, no underlying position and no existing option inventory, which is an *atypical* situation. In other circumstances, position effects (risks) can play an important, even dominant role.

Option pricing in complete markets can be viewed from several different angles. Unfortunately, the position independent perspective cannot be generalized to incomplete markets. The unifying concept for option pricing in both complete and incomplete markets is the local-equilibrium principle [5]. In light of the fact that risks of incomplete markets can be dealt with in a systematic way, insisting on risk neutral pricing is missing the spirit of using incomplete market models.

References

[1] Fischer Black (1989): "How to Use the Holes in Black-Scholes", *Journal of Applied Corporate Finance*, **1** No. 4 (Winter): 67-73

[2] Dennis Yang (2006): *Quantitative Strategies for Derivatives Trading*, ATMIF, New Jersey. Excerpt available at www.atmif.com/qsdt

[3] Dennis Yang (2006): "Derivatives Pricing and Trading in Incomplete Markets: a Tutorial on Concepts", www.atmif.com/papers

[4] Dennis Yang (2006): "A Simple Jump to Default Model", www.atmif.com/papers

[5] Dennis Yang (2007): "Beyond Black-Scholes Option Theory", www.atmif.com/papers