Derivatives Pricing and Trading in Incomplete Markets

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Outline

Describe THE Idea

The Risky Bond Example

Incomplete Market Models

Model:

- Abstraction of reality
- Simulated option game
- No absolute correctness in finance

What are the logical consequences after establishing a belief?

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Incomplete Markets:
Cannot eliminate risks associated with a derivative position.

Causes for Incompleteness:
Transaction costs, Stochastic vloatility, Jumps, Trading contraints, etc.

Reality is much better represented by incomplete markets.

Preference Question

Why is it necessary?

- The final wealth is a random variable.
- Different strategies (*e.g.* hedging schemes) produce different probability density functions of the final wealth.
- Must find a way to rank different strategies.

Example:
Strategy A: a Gaussian with mean 1.0, standard deviation 1.0;
Strategy B: a Gaussian with mean 0.5, standard deviation 0.4.
Which one do you choose?

Utility Function

Standard approach is the expected utility theory

$$E[U] = \int U(w)\rho(w)\, dw$$

Change $\int$ to $\sum$ if w is discrete.

$U(w)$ is increasing and concave.
Affine transformation freedom of utility functions.

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Use the negative exponential utility function

$$U(w) = -\frac{1}{\gamma}\exp(-\gamma w)$$

Large risk aversion parameter γ means more risk averse.
γ and position size appear together as a product.

Reason: Memoryless, Solvable

Fair Value

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How to use your fair value f:

if $p < f$, you buy;
if $p = f$, you hold;
if $p > f$, you sell;

where p is the market price of the derivative.

Review

The “Aha!” moment is coming up soon.

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- Logic

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Four ingredients:

- Logic
- Incomplete market model
- Utility function
- Notion of fair value

Aha!

In a local equilibrium when $p = f$.

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What are the necessary conditions for optimality?

$\Longrightarrow$ Equations for computing the fair value

New Pricing Principle

Local Equilibrium Principle $>$ Arbitrage Principle

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	Local equilibrium pricing	Arbitrage pricing
Complete	delta hedging & BS eq.	delta hedging $\Rightarrow$ BS eq.
Incomplete	unique and correct	a very wide range

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Explicit link: Real measure $\longrightarrow$ Pricing measure

Warning: No more freedom to yank a “risk neutral” measure out of thin air, *i.e.* cannot model “risk neutral” measure directly.

Outline

Model

- unit face value zero-coupon bond maturing at time T
- probability of default is d
- zero interest rate and other idealized assumptions
- current market price of the illiquid risky bond is p

This is an incomplete market model.

The risky bond is considered as a derivative here.

This simplest financial model goes a long way to explain all the relevant concepts.

Goal: systematic trading decisions based on the model

Portfolio Optimization

The expected utility of the final wealth is

$$E[U] = (1-d)\ U(w_0 + (1-p)\hat{n}) + d\ U(w_0 - p\hat{n})$$

Set the first order derivative w.r.t. $\hat{n}$ to zero

$$(1-d)(1-p)\ U'(w_0 + (1-p)\hat{n}) = dp\ U'(w_0 - p\hat{n})$$

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The optimal position is (no w_0)

$$\gamma\hat{n} = \ln\frac{(1-d)(1-p)}{dp}$$

Fair Value

Let n be your current position, your fair value of the risky bond is

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Inversion:

What market price makes the current position optimal?

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f depends on the model parameter d—no surprise.
f also depends on your risk preference γ and current position n!

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The fair value concept is only meaningful when you take the personal rather than the market perspective.

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Q: What is the source of the risk?
A: Having a position (your position!).

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Missing Position Dependency = Missing Risks

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Missing Position Dependency $=$ Missing Risks

The position effect can offer natural explanations to many real world phenomenons.

How to Trade

Position dependency $f(n) \Rightarrow$ Natural trading strategy

Trading Rule: (do not require gut feelings)
Make post-trade fair value equal the market price

$$f(n+m) = p$$

This is the local equilibrium equation.

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The solution is

$$m = \frac{1}{\gamma} \ln \frac{(1-d)(1-p)}{dp} - n = \hat{n} - n$$

The optimal trading size m is simply the optimal position $\hat{n}$ (post-trade) minus the current position n (pre-trade).

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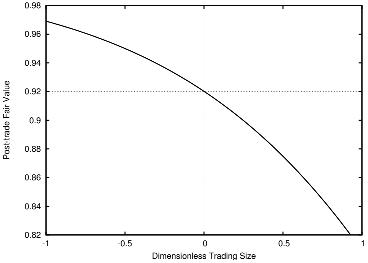
Incomplete Market Model + Risk Aversion = How to Trade

Personal Supply-Demand Curve



$d = 0.05$, $\gamma n = 0.5$

Define a curve $q(m) := f(n + m)$

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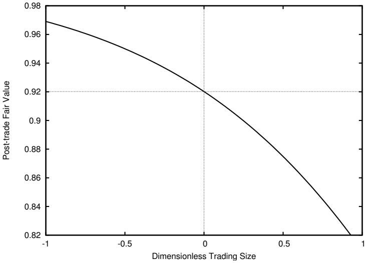

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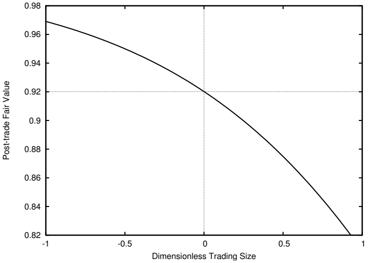

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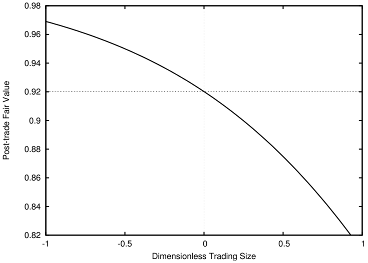

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downward sloping guarantees
equilibrium state
automatic inventory control

Generating Quotes

The personal supply-demand curve is also called quote price curve.

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Let $m_b > 0$ (bid) and $m_a < 0$ (ask)

Making a market: Posting four numbers

$\{q(m_b), |m_b|\}$—$\{q(m_a), |m_a|\}$, *e.g.*, $\{0.875, 0.5\}$—$\{0.950, 0.5\}$

{bid price, bid size}—{ask price, ask size}

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Natural market maker!

Arbitrage Price

Definition for buy and sell arbitrage prices (Why?)

$$
\begin{aligned}
a^b &:= \lim_{m \to +\infty} f(n+m) \\
a^s &:= \lim_{m \to -\infty} f(n+m)
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a^b and a^s are position and preference independent.

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a^b and a^s are position and preference independent.

Arbitrage prices are not useful in incomplete markets because (a^b, a^s) form a wide range.

For the risky bond, $a^b = 0$ and $a^s = 1$.

Certainty Equivalent Profit and Loss (CEPL)

How to measure a trade?

- Realized P&L: a random ex-post quantity
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- CEPL: convert expected utility gain into wealth

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Trading m units at p per bond:

$$E_1[U] = (1-d)\; U(w_0 - pm + n + m) + d\; U(w_0 - pm)$$

Taking the lump sum Υ in lieu of the trade:

$$E_2[U] = (1-d)\; U(w_0 + \Upsilon + n) + d\; U(w_0 + \Upsilon)$$

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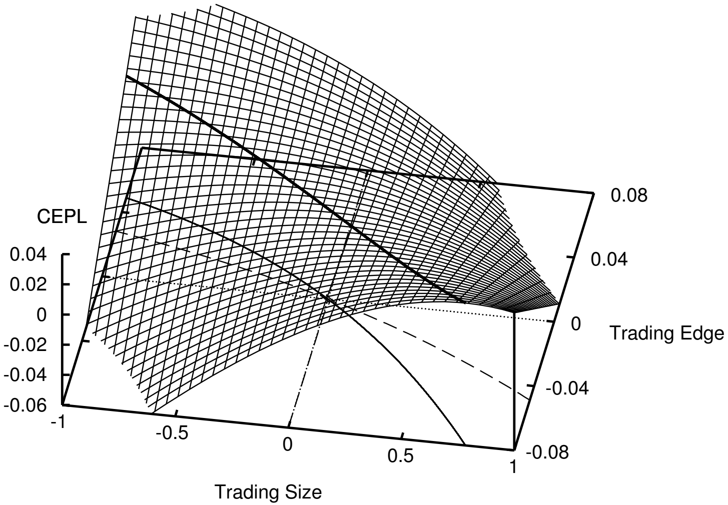
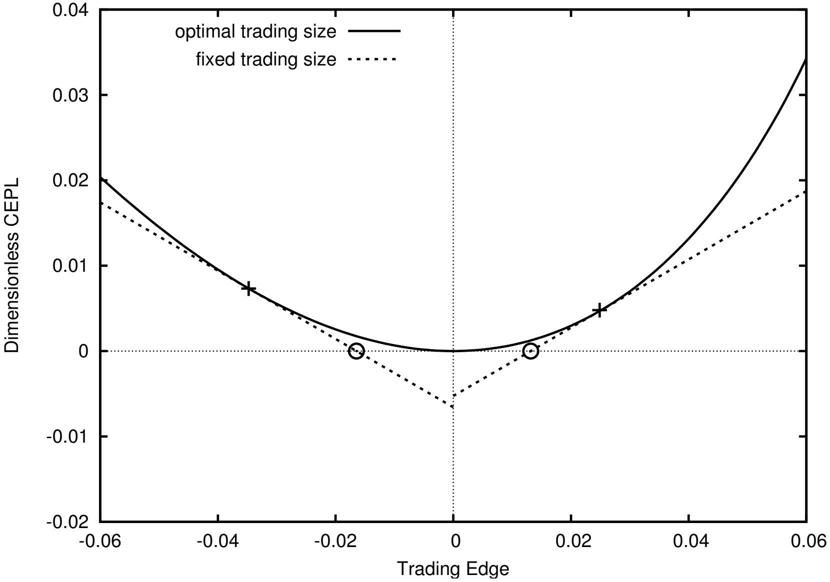
CEPL definition: Indifferent $\Rightarrow E_1[U] = E_2[U]$

$$\Upsilon(m, p) = -\frac{1}{\gamma} \ln \frac{d + (1-d)\exp[-\gamma(m+n)]}{d + (1-d)\exp(-\gamma n)} - mp$$

Dimensionless CEPL Surface $\gamma\Upsilon(m, p)$

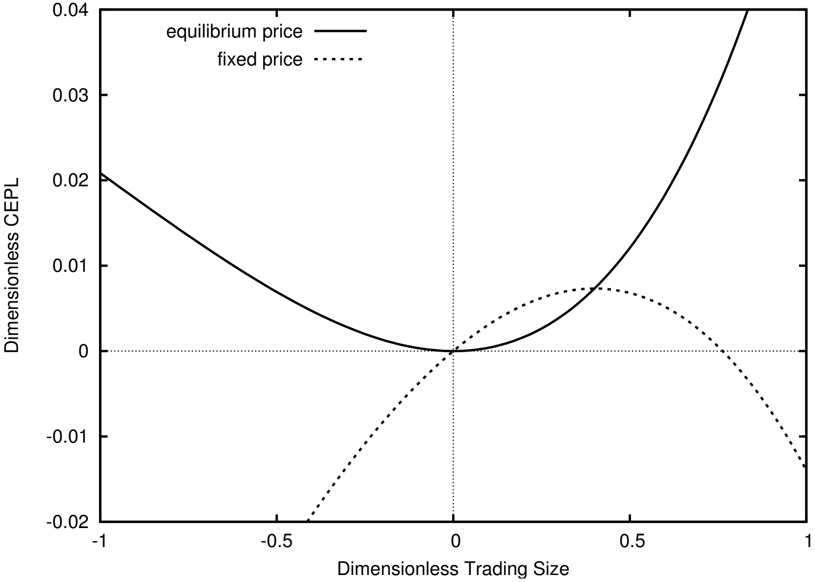


CEPL against Trading Price



Sideway view of the surface plot

CEPL against Trading Size



Front view of the surface plot

Portfolio Indifference Price

Indifferent between lump sum h and position n

$$U(w_0 + h) = (1 - d)\ U(w_0 + n) + d\ U(w_0)$$

Explicit formula

$$h(n) = -\frac{1}{\gamma} \ln\left[d + (1 - d) \exp(-\gamma n)\right]$$

Note $n = 0 \Rightarrow h = 0$.

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The CEPL formula can be rewritten as

$$\Upsilon(m, p) = h(m + n) - h(n) - mp$$

Can be deduced from the notion of indifference.

Tangent Relation

$$f(n) = h'(n)$$

Easy proof mathematically

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Two proofs based on financial interpretations:

Proof #1: Infinitesimal trade after establishing equilibrium

$$0 = h(\epsilon + n) - h(n) - \epsilon p$$

Tangent Relation

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Proof #1: Infinitesimal trade after establishing equilibrium

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Concavity of $h(n) \Rightarrow$ downward slope of $f(n)$

Reserve Price

Why needed? Trading size not infinitely divisible.

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Zero CEPL if trading m units at $r(m)$ per unit

$$
\begin{aligned}
r(m) &= \frac{1}{m}\left[h(n+m) - h(n)\right] \\
&= \frac{1}{\gamma m} \ln \frac{d + (1-d)\exp(-\gamma n)}{d + (1-d)\exp[-\gamma(n+m)]}
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Another CEPL formula: (easy financial interpretation)

$$\Upsilon(m, p) = m\,[r(m) - p]$$

Negative CEPL if $r^b(|m|) < p < r^s(|m|)$.

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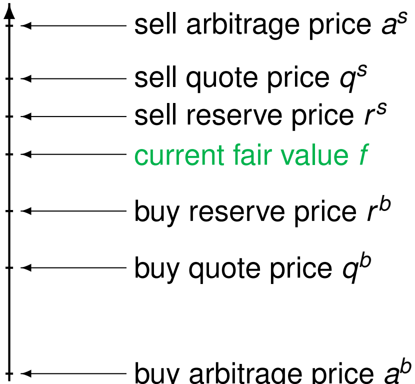
Optimal CEPL formula:

$$\Upsilon_o(m) = m\left[r(m) - q(m)\right] \geq 0$$

$\Upsilon_o(m) \leftarrow$ quote price curve $q(m) \rightarrow \Upsilon_o(p - f(n))$

Schematic Drawing



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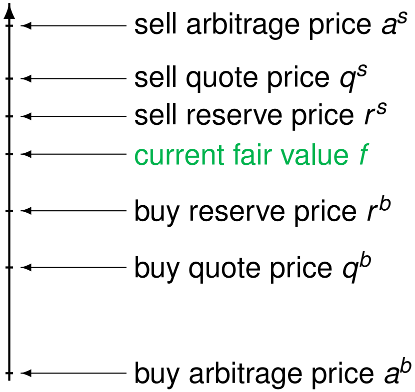

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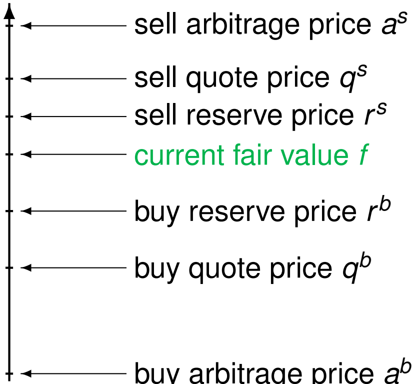

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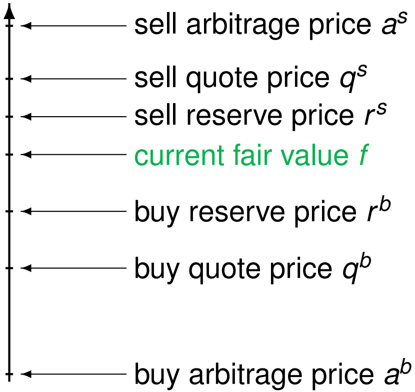
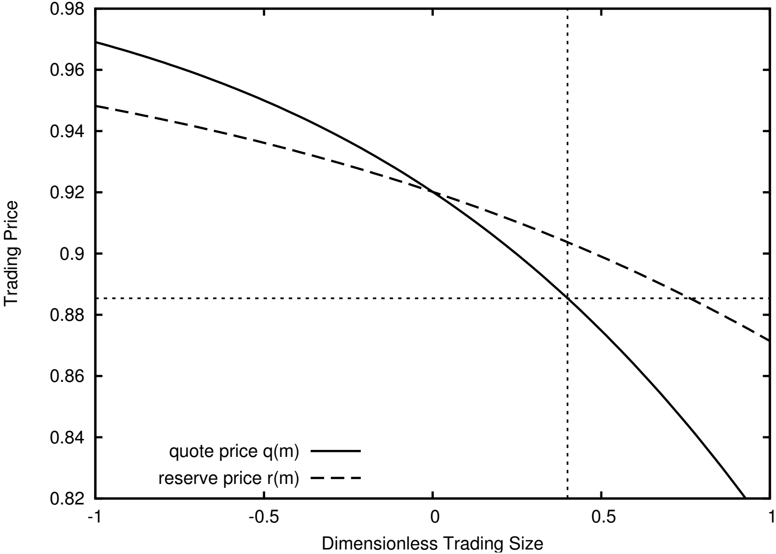


- Meaning w.r.t. trading size
- Intuitive ranking
- $q(m)$ and $r(m)$ asymmetric w.r.t. current fair value
- $r(m) \approx \frac{1}{2}[q(m) + q(0)]$

Quote and Reserve Price Curves



Basis for making rational trading decisions!

Mutually Beneficial Trading

Example: Same everything except initial position

	γn	c.f.v.	γm	p.t.f.v	$\gamma \Upsilon$
Trader A	0.0	0.9500	0.25	0.9367	1.724×10^{-3}
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Economical Reason: Risk Transfer!

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Global Equilibrium: There exists a global equilibrium state for M traders.

May not reach there in a reasonable amount of time!

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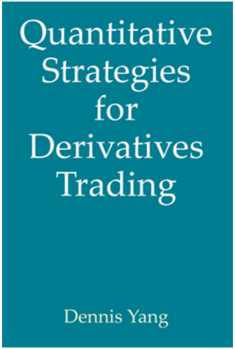
- derivatives pricing is preference and position dependent in incomplete markets, which is only meaningful from the personal perspective;
- the position dependent pricing offers a natural and systematic way to trade derivatives;
- derivatives trading in incomplete markets is mutually beneficial.

Further Information: www.atmif.com/qsdt



- Book Excerpt
- Derivatives Pricing and Trading in Incomplete Markets: A Tutorial on Concepts
- A Simple Jump to Default Model